Abstract

This MAPLE worksheet is concerned with the deflection curve of a statically indeterminate beam rigidly clamped at both ends. The beam may be loaded by a concentrated force \( F \) in vertical and a load \( L \) in longitudinal direction. The elastic foundation is characterised by the parameter \( K \). This problem can easily solved by applying the LAPLACE transformation to the differential equation of the problem.

Keywords: Elastic foundation; Deflection of beams; LAPLACE transformation

Differential Equation

The deflection curve \( y(x) \) of a beam on an elastic foundation is the solution of the following ordinary differential equation with constant coefficients:

\[
\frac{d^4}{dx^4}y(x) + L\frac{d^2}{dx^2}y(x) + Ky(x) + Q(x) = 0
\]

The parameters \( L \) and \( K \) are expressing the longitudinal loading and the elastic foundation, respectively, where \( EI \) is the flexural rigidity and beta is known as the WINKLER elastic foundation number. The vertical load \( Q(x) \) is:

\[
Q(x) := \frac{q(x)}{EI}
\]

In the following example \( Q(x) \) is assumed to be a concentrated force \( F = A*EI \).
at \( x = a \), which can be expressed by the \textit{DIRAC delta function}. Thus, we start from the following differential equation:

\[
\text{dgl\_eqn} := \left( \frac{d^4}{dx^4} y(x) \right) + L \left( \frac{d^2}{dx^2} y(x) \right) + K y(x) = A \delta(x - a)
\]

\textbf{Boundary Conditions}

To calculate the deflection curve of a statically indeterminate beam rigidly clamped at both ends, we have to take the following boundary conditions into consideration:

\[
\begin{align*}
C[3] &:= \frac{d^3}{dx^3} y(x) \bigg|_{x=0}; \quad C[2] := \frac{d^2}{dx^2} y(x) \bigg|_{x=0}; \\
C[1] &:= \frac{d}{dx} y(x) \bigg|_{x=0}; \quad C[0] := y(x) \bigg|_{x=0}
\end{align*}
\]


\textbf{LAPLACE Transformation}

Using the MAPLE \texttt{inttrans} package we arrive at the following solution:

\[
\text{laplace}(\text{dgl\_eqn}, x, s) \quad \text{assuming} \quad a \geq 0;
\]

\[
s^4 \text{laplace}(y(x), x, s) - (D^{(3)})(y)(0) - s \left( D^{(2)})(y)(0) - s^2 D(y)(0) - s^3 y(0) \right) + L s^2 \text{laplace}(y(x), x, s) - L D(y)(0) - L s y(0) + K \text{laplace}(y(x), x, s) = A e^{-s a}
\]

\[
\text{subs}((D(D(D(y))))(0) = C[3], D(D(y))(0) = C[2], D(y)(0) = C[1], y(0) = C[0]), \%);
\]

\[
s^4 \text{laplace}(y(x), x, s) - \left( \frac{d^3}{dx^3} y(x) \right)(0) - s \left( \frac{d^2}{dx^2} y(x) \right)(0) - s^2 \left( \frac{d}{dx} y(x) \right)(0) - s^3 y(0) + L s^2 \text{laplace}(y(x), x, s) - L s y(0) + K \text{laplace}(y(x), x, s) = A e^{-s a}
\]

\[
\]

\[
s^4 \text{laplace}(y(x), x, s) - B_3 - s B_2 + L s^2 \text{laplace}(y(x), x, s) + K \text{laplace}(y(x), x, s) = A e^{-s a}
\]

\[
\text{readlib(isolate)}(\%); \quad \text{laplace}(y(x), x, s) = \frac{A e^{-s a} + B_3 + s B_2}{s^4 + L s^2 + K}
\]
Note that the commands of the solution \( Y(x) \) are executed but the outputs are not printed, since the input lines are ending by colons instead of semicolons. The constants \( B_2 \) and \( B_3 \) can be determined by considering the boundary conditions at \( x = 1 \):

\[
Y(1) := 0.02342 \times 10^{-3} + 0.2471 \times 10^{-3} \times B_2 + 0.8892 \times 10^{-3} \times B_3
\]

\[
T(1) := 0.1510 \times 10^{-3} + 0.8892 \times 10^{-3} \times B_4 + 2.442 \times 10^{-3} \times B_2
\]

\( \text{evalf} \left( \text{solve} \left( \{Y(1)=0, T(1)=0\}, \{B_2, B_3\} \right) \right) \)

\[
\{B_2 = 0.088044, B_3 = -0.41161\}
\]

Again this solution is not printed, since the input command is ended by a colon instead of a semicolon.

Representation with rational numbers:

\[
z(x) := \text{convert} \left( Z(x), \text{`rational`} \right)
\]

\[
z(x) := \text{simplify} \left( \right)
\]

\( \text{plots[display]} \left( \text{plot1, plot2, plot3} \right) \)

Deflection # parameters: \( a = 1/2, A = 1, L = -10, K = 100 \)
The concentrated force $F = A*EI$ at $x = a$ can be expressed by the \textit{DIRAC} delta function, mentioned before.

Alternatively, we can use the \textit{HEAVISIDE} function:

$$F(x,a) := H(x-(a-\epsilon))-H(x-(a+\epsilon));$$

Concentrated force at $a = 1/2 - \epsilon$ and $a = 1/2 + \epsilon$:

$$F(x,0.5) := \text{subs(} \{a=0.5, \epsilon=0.001\}, \%);$$

$$F\left(\frac{1}{2}\right) := \text{evalf}\left(\frac{499}{1000} - \frac{501}{1000}\right);$$

Maximum deflection at $a = 1/2$ with $A = 1$, $L = -10$, $K = 100$:

$$Z_{\text{max}} := \text{evalf}\left(\text{subs(} x=1/2.001, z(x)\), 4\right);$$

$$Z_{\text{max}} := 0.003642$$

$$Z_{\text{max}} := \text{evalf}\left(\text{subs(} x=1.001/2, z(x)\), 4\right);$$

$$Z_{\text{max}} := 0.003629$$

$$Z_{\text{max}} := \text{evalf}\left(\left(\frac{Z_{\text{max}}+Z_{\text{max}}}{2}\right), 4\right);$$

$$Z_{\text{max}} := 0.003635$$

The dimensionless maximum of the deflection of a beam with $L = K = 0$ is $1/192$, as shown in: BETTEN, J., \textit{Creep Mechanics}, Third Edition, Springer-Verlag, Berlin / Heidelberg / New York, 2008. This value is compared with the corresponding value.
for $L$ and $K$ not equal to zero in the following:

\[
\text{relative\_maximum\_deflection} := \frac{z_{\text{max}}[K=0,L=0] - z_{\text{max}}[K,L]}{z_{\text{max}}[K=0,L=0]};
\]

For $K = 100$ and $L = -10$ we arrive at:

\[
\frac{z_{\text{max}}[K=0,L=0] - z_{\text{max}}[K=100,L=-10]}{z_{\text{max}}[K=0,L=0]} = \text{evalf}((1/192-0.003635)*192), 4);
\]

\[
\text{convert}(%,'\text{rational}') ;
\]

\[
\frac{z_{\text{max}}[K=0,L=0] - z_{\text{max}}[K=100,L=-10]}{z_{\text{max}}[K=0,L=0]} = \frac{3021}{10000}
\]

The length of the beam may be 10 m. Then, we arrive for $L = -10$ and $K = 100$ at a deflection maximum of 3.64 cm. The corresponding value for $L = K = 0$ is $(1/192)*10^3$ cm = 5.21 cm. The quotient is:

\[
\text{quotient} := \text{evalf}(z_{\text{max}}[K=0,L=0]/z_{\text{max}}[K=100,L=-10]=(1/192)/Z_{\text{max}}), 4);
\]

\[
\text{quotient} := \frac{z_{\text{max}}[K=0,L=0]}{z_{\text{max}}[K=100,L=-10]} = 1.435
\]

Another example: assuming $K = 100$ and $L = 0$; then, we arrive at a quotient of:

\[
\text{Quotient} := \text{evalf}(z_{\text{max}}[K=100,L=-10]/z_{\text{max}}[K=100,L=0] = 0.003635/0.004367, 4);
\]

\[
\text{Quotient} := \frac{z_{\text{max}}[K=100,L=-10]}{z_{\text{max}}[K=100,L=0]} = 0.8324
\]

This value demonstrates the influence of a longitudinal force ($L = -10$) on the deflection maximum of a beam on an elastic foundation, i.e., because of $L$ the deflection decreases. An increasing positive parameter $L$ leads to a critical load in horizontal direction.