
D1 Nonlinear Flight-Mechanics Models and their Linearisation

D1.1 Introduction

The equations of motion of a helicopter can be written in nonlinear form as:

$$\dot{x} = f(x, u, t)$$  \hfill (D1.1)

where $x$ represents the relevant state variables of the model and $u$ represents the inputs. In the general case $x$ would include both state variables associated with rigid body motion of the fuselage and state variables associated with the rotors.

For experimental modelling using system identification and parameter estimation most published work involves six-degree-of-freedom descriptions. Similarly, for the purposes of initial modelling of vehicles for flight control system design, much work has been carried out using six-degree-of-freedom models. In such cases, using the symbols normally adopted for aircraft flight mechanics modelling the vector of state variables is:

$$x = [u, w, q, \theta, p, \phi, r, \psi]^T$$  \hfill (D1.2)

Here the variables $u$, $v$ and $w$ are the three translational velocities measured with respect to a fuselage-fixed set of axes and $p$, $q$ and $r$ are the angular velocities about that $x$, $y$ and $z$ axis system. The angles $\theta$, $\phi$ and $\psi$ are the Euler angles and define the orientation of the body axis system relative to the earth.

The input vector is:

$$u = [\eta_0, \eta_{1s}, \eta_{1c}, \eta_{tr}]^T$$  \hfill (D1.3)

and involves the variable $\eta_0$ which is the main rotor collective input, $\eta_{1s}$ which is the main rotor longitudinal cyclic input, $\eta_{1c}$ which is the main rotor lateral cyclic input and $\eta_{tr}$ which is the pedal input to control the tail rotor collective pitch.

The set of nonlinear equations for this system has the general form (see e.g. [1]):

$$\dot{u} = -(wq - vr) - g \sin \theta + \frac{x}{m}$$  \hfill (D1.4)

$$\dot{v} = -(ur - wp) + g \cos \theta \sin \phi + \frac{y}{m}$$  \hfill (D1.5)

$$\dot{w} = -(vp - uq) + g \cos \theta \cos \phi + \frac{z}{m}$$  \hfill (D1.6)
\[ \dot{p} = \frac{(l_{yy} - l_{xz})}{l_{xx}} qr + \frac{l_{xz}}{l_{xx}} (\dot{r} + pq) + \frac{L}{l_{xx}} \]  

(D1.7)

\[ \dot{q} = \frac{(l_{xz} - l_{xx})}{l_{yy}} rp + \frac{l_{xx}}{l_{yy}} (r^2 - p^2) + \frac{M}{l_{yy}} \]  

(D1.8)

\[ \dot{r} = \frac{(l_{xx} - l_{yy})}{l_{zz}} pq + \frac{l_{xz}}{l_{zz}} (\dot{p} - qr) + \frac{N}{l_{zz}} \]  

(D1.9)

\[ \dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \]  

(D1.10)

\[ \dot{\theta} = q \cos \phi - r \sin \phi \]  

(D1.11)

\[ \psi = q \sin \phi \sec \theta + r \cos \phi \sec \theta \]  

(D1.12)

where \( m \) is the mass of the vehicle, \( l_{xx}, l_{yy} \) and \( l_{zz} \) are the moments of inertia of the vehicle about the reference axes and \( l_{xz} \) is the roll/yaw product of inertia. The external forces \((X, Y \text{ and } Z)\) and external moments \((L, M \text{ and } N)\) include contributions (where appropriate) from the main rotor, tail rotor, fuselage, horizontal tail plane and vertical fin. Further details of models of this form may be found in many books on helicopter flight mechanics (e.g. [1]).

### D1.2 Linearisation of the Equations of Motion

In developing linearised models of helicopter motion it is normal to assume that we are considering perturbations from a trim (equilibrium) state. A trimmed flight condition is defined as one in which the rate of change of the state vector is zero giving

\[ f(x_e, u_e) = 0 \]  

(D1.13)

where the inputs \( u_e \) from the four controls (main rotor collective pitch, main rotor longitudinal cyclic pitch, main rotor lateral cyclic pitch and tail rotor collective pitch) provide the forces and moments required to hold a defined steady state \( x_e \). Clearly, with four inputs only four states can be prescribed in the trimmed condition.

Describing the motion of the vehicle as a perturbation from the trim state we have:

\[ x = x_e + \delta x \]  

(D1.14)

Each of the six external forces and moments \((X, Y, Z, L, M \text{ and } N)\) can then be expressed, using Taylor’s theorem, in an approximate form by neglecting all terms except the linear terms. The six forces and moments can then each be written in an approximate form typified by:

\[ X = X_e + \frac{\partial X}{\partial u} \delta u + \frac{\partial X}{\partial v} \delta v + \cdots + \frac{\partial X}{\partial \theta_0} \delta \theta_0 + \cdots \text{ etc.} \]  

(D1.15)

It is conventional to write the partial derivatives in a more compact form so that \( \frac{\partial X}{\partial u} \) is written as \( X_u \) and other derivatives are expressed in a similar fashion as \( M_v, L_{\theta_0}, \text{etc.} \) The linearised equations can then be written as:
\[
\dot{x} = Ax + Bu + h
\]  
(D1.16)

where a function \( h(t) \) has been added to represent atmospheric and other disturbances. The elements of the \( A \) and \( B \) matrices may be derived from the nonlinear function \( f(x, u, t) \) that appears in (D1.1). Hence:

\[
A = \left( \frac{\partial f}{\partial x} \right)_{x=x_e}
\]  
(D1.17)

\[
B = \left( \frac{\partial f}{\partial u} \right)_{x=x_e}
\]  
(D1.18)

It is normal practice to omit the heading angle \( \psi \) from the set of state variables since the direction of flight in the horizontal plane has no effect on the forces and moments.

Analytic methods can, in principle, be used to find the values of the coefficients within the \( A \) and \( B \) matrices but it is normal to use a numerical type of approach based on numerical differencing. Although this is approximate, numerical derivatives converge to the analytical values for perturbations that are sufficiently small. The elements within the \( A \) and \( B \) matrices are traditionally called “stability and control derivatives”, respectively. They are denoted using the form \( X_u, M_q, N_{tr} \) etc where, for example, \( \frac{\partial X}{\partial \xi}, \frac{\partial M}{\partial \xi} \) etc, where the partial derivatives are all evaluated at the trimmed state \( x = x_e \).

If we re-arrange the elements of the state vector in the linearised model we can restructure the complete \( 8 \times 8 \) system matrix so that longitudinal elements are grouped together as a \( 4 \times 4 \) sub-matrix, the lateral elements form a further \( 4 \times 4 \) sub-matrix and there are two further \( 4 \times 4 \) submatrices representing coupling between the longitudinal and lateral sub-systems. Hence if, as before,

\[
x = [u \ w \ q \ \theta \ v \ p \ \phi \ r]^T
\]

and \( u = [\eta_0 \ \eta_{1s} \ \eta_{1c} \ \eta_{tr}]^T \)

the system matrix \( A \) within the linearised model of (D1.16) can be written in the form:

\[
A = \begin{bmatrix}
X_u & X_w & X_q & X_\theta & X_v & X_p & X_\phi & X_r \\
Z_u & Z_w & Z_q & Z_\theta & Z_v & Z_p & Z_\phi & Z_r \\
M_u & M_w & M_q & M_\theta & M_v & M_p & M_\phi & M_r \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
Y_u & Y_w & Y_q & Y_\theta & Y_v & Y_p & Y_\phi & Y_r \\
L_u & L_w & L_q & L_\theta & L_v & L_p & L_\phi & L_r \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
N_u & N_w & N_q & N_\theta & N_v & N_p & N_\phi & N_r
\end{bmatrix}
\]  
(D1.19)

and the input matrix \( B \) is:

\[
B = \begin{bmatrix}
X_0 & X_{1s} & X_{1c} & X_{tr} \\
Z_0 & Z_{1s} & Z_{1c} & Z_{tr} \\
M_0 & M_{1s} & M_{1c} & M_{tr} \\
0 & 0 & 0 & 0 \\
Y_0 & Y_{1s} & Y_{1c} & Y_{tr} \\
L_0 & L_{1s} & L_{1c} & L_{otr} \\
0 & 0 & 0 & 0 \\
N_0 & N_{1s} & N_{1c} & N_{tr}
\end{bmatrix}
\]  
(D1.20)
The derivatives appearing as the elements within the $A$ and $B$ matrices can either be found on a theoretical basis from the underlying nonlinear model or can be estimated from flight data using system identification and parameter estimation methods. Comparison of the theoretical values and the values estimated from flight data provides a basis for a form of model testing and validation, as outlined in Chapters 5 and 7 and discussed further in Chapter 10 in the helicopter case study.

It should be noted that the determination of the theoretical stability and control derivatives can be performed analytically at the trimmed state but it is more conventional to estimate these values using a numerical process based on finite differencing for specific sizes of perturbation. The size of perturbation applied can be important when strong nonlinearities are known to exist in the region of the trimmed state. Issues of this kind are discussed in detail by Padfield [1].

It should be noted that the nonlinear flight mechanics model presented in equations (D1.4) – (D1.12) does not involve any state variables associated with the dynamics of the rotors. It is therefore a simplified description and when stability and control derivatives of this simplified model are estimated from flight data using system identification methods dynamic effects from the rotor will influence the parameter values found for the simple six-degrees-of-freedom representation of equations (D1.16) – (D1.20). A “folding” phenomenon [1] occurs, through which the effects of rotor degrees of freedom are included within the parameters of the fuselage model. The resulting simple models are only valid if the relevant characteristic frequencies of the fuselage and rotor are well separated and the closer these characteristic frequencies become the more necessary it is to include these additional state variables associated with the dynamics of the rotor (see e.g. [1] - [3]). Current practice in terms of flight mechanics models used for flight control system design, especially for helicopters equipped with modern hinge-less rotors, is to include state variables associated with rotor flapping dynamics, lead-lag dynamics and inflow dynamics [4]. Estimation of parameters associated with rotor state variables remains challenging because of inherent difficulties associated with measurement of these variables in flight. The use of frequency-domain methods of system identification can help to ensure that parameters of six-degree-of-freedom models are not significantly affected by higher frequency modes associated with the rotor by limiting the range of frequencies considered in the identification process. Further details of this frequency-domain approach may be found elsewhere (see e.g. [5]).

**Example 1.**

**Puma SA330 Helicopter** (a transport helicopter in the 6-tonne class) for a flight condition involving 80 knots level forward flight.

In this case some elements of the $A$ and $B$ matrices are well known as they depend on physical quantities such as the trimmed forward velocity and the gravitational constant. Some others are known to be insignificant in terms of their effects on the overall system dynamics and can be set to zero [6]. The model of (D1.19) and (D1.20) can therefore be simplified to give the matrices shown in (D1.21) and (D1.22) where the vector of state variables $x = [u \ v \ q \ \theta \ \eta_\phi \ p \ q \ r]^T$ where units are ms$^{-1}$ for velocities $u$, $v$ and $w$; radians for angles $\theta$ and $\phi$ and rad s$^{-1}$ for the angular rates $p$, $q$ and $r$. The input vector is $u = [\eta_0 \ \eta_{1s} \ \eta_{1c} \ \eta_{ef}]^T$ where the units are radians.
One important point that emerges from examination of the $A$ matrix above is that there is evidence of coupling between the longitudinal and lateral state variables, due to the parameters $M_v$, $M_p$, $L_u$, $L_w$, $L_q$, and $N_w$. Compared with most fixed-wing aircraft models this coupling is a potential source of difficulty in carrying out system identification and parameter estimation from data collected during helicopter flight tests. Although simplifications have been made, the number of parameters to be estimated is higher than in the case of many types of fixed-wing aircraft where identification of longitudinal and lateral sub-models can be carried out separately. Such separation is not possible for helicopter system identification, although some simplification can be achieved by regarding lateral responses as “pseudo-control inputs” to a longitudinal model and vice versa [7].

The theoretical values for other parameters (using the numerical values reported in [7], as found using HELISTAB/Helisim flight mechanics modelling software [6] ) are shown in the matrices below:

$$A = \begin{bmatrix}
x_u & x_w & 0 & -9.81 & 0 & 0 & 0 & 0 \\
z_u & z_w & 41.0 & 0 & 0 & 0 & 0 & 0 \\
m_u & m_w & m_q & 0 & m_v & m_p & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & y_v & y_p & 9.81 & -41.0 \\
l_u & l_w & l_q & 0 & l_v & l_p & 0 & l_r \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
N_u & N_w & N_q & 0 & N_v & N_p & 0 & N_r \\
\end{bmatrix}$$  \hspace{1cm} (D1.21)

$$B = \begin{bmatrix}
x_0 & x_{1s} & 0 & 0 \\
z_0 & z_{1s} & 0 & 0 \\
m_0 & m_{1s} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & y_{1c} & y_{tr} \\
0 & 0 & l_{1c} & l_{tr} \\
0 & 0 & 0 & 0 \\
0 & 0 & N_{1c} & N_{tr} \\
\end{bmatrix}$$  \hspace{1cm} (D1.22)

$$A = \begin{bmatrix}
-0.024 & 0.002 & 0 & -9.81 & 0 & 0 & 0 & 0 \\
-0.048 & -0.730 & 41.0 & 0 & 0 & 0 & 0 & 0 \\
0.007 & -0.020 & -0.766 & 0 & -0.005 & -0.21 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.125 & 0.751 & 9.81 & -41.0 \\
-0.006 & -0.053 & 0.758 & 0 & -0.055 & -1.677 & 0 & 0.142 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0.033 & 0 & 0 & 0.022 & -0.174 & 0 & -0.569 \\
\end{bmatrix}$$  \hspace{1cm} (D1.23)
Example 2. Westland Lynx Helicopter for hovering flight. In this case the state vector is (as before) $x = [u \ w \ q \ \theta \ v \ p \ \phi \ r]^T$ but the input vector $u = [\theta_0 \ \theta_{1s} \ \theta_{1c} \ \theta_{tr}]^T$ where $\theta_0$ is the main rotor collective pitch angle (at the rotor itself rather than at the pilot’s collective pitch control input lever), $\theta_{1s}$ is the main rotor longitudinal cyclic pitch angle, $\theta_{1c}$ is the main rotor lateral cyclic pitch angle and $\theta_{tr}$ is the tail rotor collective pitch angle. The structure of the model has not, in this case, been simplified by eliminating any of the derivatives. The data are as reported in [8], as found using HELISTAB/Helisim flight mechanics modelling software [6]). The $A$ and $B$ matrices are:

$$A = \begin{bmatrix} -0.025 & 0.022 & 0.668 & -9.78 & -0.021 & -0.160 & 0.000 & 0.000 \\ 0.028 & -0.312 & 0.013 & -0.722 & -0.003 & -0.005 & 0.521 & 0.000 \\ 0.048 & 0.0051 & -1.896 & 0.000 & 0.059 & 0.456 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.999 & 0.000 & 0.000 & 0.000 & 0.000 & 0.053 \\ 0.021 & 0.000 & -0.161 & 0.038 & -0.043 & -0.689 & 9.77 & 0.137 \\ 0.340 & 0.024 & -2.645 & 0.000 & -0.270 & -10.975 & 0.000 & -0.028 \\ 0.000 & 0.000 & -0.004 & 0.000 & 0.000 & 1.000 & 0.000 & 0.074 \\ 0.061 & 0.009 & -0.478 & 0.000 & 0.002 & -1.924 & 0.000 & -0.374 \end{bmatrix} \quad (D1.23)$$

$$B = \begin{bmatrix} 6.942 & -9.286 & 2.016 & 0.000 \\ -93.92 & -0.002 & 0.000 & 0.000 \\ 0.955 & 26.40 & -5.733 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ -0.357 & -2.016 & -9.286 & 5.615 \\ 7.048 & -33.212 & -160.0 & -1.124 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ -17.31 & -5.991 & -27.59 & -15.14 \end{bmatrix} \quad (D1.24)$$

Example 3. Westland Lynx Helicopter for a flight condition involving level forward flight at 80 knots. The state vector is (as before) $x = [u \ w \ q \ \theta \ v \ p \ \phi \ r]^T$ and the input vector $u = [\theta_0 \ \theta_{1s} \ \theta_{1c} \ \theta_{tr}]^T$. The structure of the model has not, in this case, been simplified by eliminating any of the derivatives. The data are as reported in [9], as found using HELISTAB/Helisim flight mechanics modelling software [6]). The $A$ and $B$ matrices are:
$$A = \begin{bmatrix}
-0.032 & 0.040 & -0.226 & -9.808 & -0.002 & -0.109 & 0.000 & 0.000 \\
-0.010 & -0.802 &  41.09 & -0.211 & -0.019 & -0.451 & 0.322 & 0.000 \\
 0.027  &  0.029 & -2.341 &  0.000 &  0.010 &  0.410 & 0.000 & 0.000 \\
 0.000  &  0.000 &  0.999 &  0.000 &  0.000 &  0.000 & 0.000 & 0.000 \\
 0.043  &  0.014 & -0.128 &  0.007 & -0.167 &  0.199 & 9.803 & -40.69 \\
-0.037  &  0.234 & -1.996 &  0.000 & -0.163 & -10.536 & 0.000 & -0.287 \\
 0.000  &  0.000 & -0.001 &  0.000 &  0.000 &  1.000 & 0.000 & 0.022 \\
-0.026  &  0.002 & -0.089 &  0.000 &  0.101 & -1.793 & 0.000 & -1.349 
\end{bmatrix} \quad \text{(D1.25)}$$

$$B = \begin{bmatrix}
4.345 & -7.633 & 2.058 & 0.000 \\
-117.79 & -30.391 & 0.000 & 0.000 \\
14.078 & 28.540 & -5.855 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 \\
1.499 & -1.528 & -9.320 & 6.704 \\
32.071 & -25.031 & -153.23 & -1.342 \\
0.000 & 0.000 & 0.000 & 0.000 \\
\end{bmatrix} \quad \text{(D1.26)}$$

References


