Solutions to Supplementary Problems

Chapter 1

Motion of a Point Mass
Example 1.17 A point $P$ moves on a given path from $A$ to $B$ (Fig. 1.42). Its velocity $v$ decreases linearly with arc-length $s$ from a value $v_0$ at $A$ to zero at $B$.

How long does it take $P$ to reach point $B$?

Solution First we write down the velocity $v$ as a linear function of the arc-length $s$:

$$v(s) = v_0 \left(1 - \frac{s}{l}\right).$$

Separation of variables

$$\frac{ds}{v_0 (1 - \frac{s}{l})} = dt$$

and indeterminate integration lead to

$$\int \frac{ds}{1 - \frac{s}{l}} = v_0 \int dt \quad \Rightarrow \quad s = l \left[1 - \exp \left(\frac{-v_0 t}{l}\right)\right].$$

The integration constant $C$ is determined from the initial condition:

$$s(0) = 0 \quad \Rightarrow \quad 0 = 1 - C \quad \Rightarrow \quad s = l \left[1 - \exp \left(\frac{-v_0 t}{l}\right)\right].$$

Point $B$ is reached when $s = l$. This yields the corresponding time $t_B$:

$$s(t_B) = l \quad \Rightarrow \quad t_B \rightarrow \infty.$$
**Example 1.18** A radar screen tracks a rocket which rises vertically with a constant acceleration \( a \) (Fig. 1.43). The rocket is launched at \( t = 0 \).

Determine the angular velocity \( \dot{\phi} \) and the angular acceleration \( \ddot{\phi} \) of the radar screen. Calculate the maximum angular velocity \( \dot{\phi} \) and the corresponding angle \( \phi \).

**Solution** Since the acceleration \( a \) is constant, the velocity \( v \) and the position \( x \) of the rocket are

\[
v = at + v(0),
\]
\[
x = at^2/2 + v(0)t + x(0).
\]

Applying the initial conditions yields

\[
v(0) = 0 \quad \rightarrow \quad v = at,
\]
\[
x(0) = 0 \quad \rightarrow \quad x = at^2/2.
\]

The angle \( \phi \) of the radar screen follows as

\[
\tan \phi = \frac{x}{l} \quad \rightarrow \quad \phi(t) = \arctan \left( \frac{at^2}{2l} \right).
\]

Differentiation leads to the angular velocity

\[
\dot{\phi}(t) = \frac{at}{l} \left[ 1 + \left( \frac{at^2}{2l} \right)^2 \right]^{-1/2}
\]

and the angular acceleration

\[
\ddot{\phi}(t) = \frac{a}{l} \left( 1 - \frac{3a^2t^4}{4l^4} \right) \left[ 1 + \left( \frac{at^2}{2l} \right)^2 \right]^{-3/2}.
\]
The time $t^*$ of the maximum angular velocity $\dot{\varphi}_{\text{max}}$ is obtained from

\[ \ddot{\varphi}(t^*) = 0 \quad \rightarrow \quad t^* = \left( \frac{4l^2}{3a^2} \right)^{1/4}. \]

Thus,\[ \dot{\varphi}_{\text{max}} = \dot{\varphi}(t^*) \quad \rightarrow \quad \dot{\varphi}_{\text{max}} = \sqrt{\frac{3\sqrt{3}a}{8l}} \]

and\[ \varphi(t^*) = \arctan(1/\sqrt{3}) \quad \rightarrow \quad \varphi(t^*) = 30^\circ. \]
Example 1.19 Two point masses $P_1$ and $P_2$ start at point $A$ with zero initial velocities and travel on a circular path. $P_1$ moves with a uniform tangential acceleration $a_{t1}$ and $P_2$ moves with a given uniform angular velocity $\omega_2$.

a) What value must $a_{t1}$ have in order for the two masses to meet at point $B$?
b) What is the angular velocity of $P_1$ at $B$?
c) What are the normal accelerations of the two masses at $B$?

Solution The velocity and position of $P_1$ follow from $a_{t1} = \text{const} = \dot{v}$ with the initial conditions $s_{01} = 0$ and $v_{01} = 0$ as

$$v_1 = a_{t1} t, \quad s_1 = \frac{1}{2} a_{t1} t^2.$$  

Similarly, for point $P_2$ we obtain from $a_{t2} = 0$ with $s_{02} = 0$ and $v_{02} = r\omega_2$:

$$v_2 = r\omega_2, \quad s_2 = r\omega_2 t.$$  

a) Both points meet at time $t_B$ at point $B$. Thus,

$$\pi r = \frac{1}{2} a_{t1} t_B^2, \quad \pi r = r\omega_2 t_B.$$  

This leads to

$$t_B = \frac{\pi}{\omega_2} \quad \Rightarrow \quad a_{t1} = \frac{2\pi r}{t_B^2} = \frac{2r\omega_2^2}{a_{t1}}.$$
b) The tangential acceleration $a_{t1}$ and the time $t_B$ are now known. Hence, we can calculate the angular velocity of $P_1$ at $B$:

\[ v_1(t_B) = a_{t1} t_B = r \omega_1(t_B) \]

\[ \rightarrow \quad \omega_1(t_B) = \frac{a_{t1} t_B}{r} = \frac{2r \omega_2^2 \pi}{\pi r \omega_2} = 2 \omega_2 \]

c) The normal accelerations at $B$ follow from $a_n = r \omega^2$:

\[ a_{n1} = r \omega_1^2(t_B) = 4r \omega_2^2 \]

\[ a_{n2} = r \omega_2^2 \]
Example 1.20  A child of mass \( m \) jumps up and down on a trampoline in a periodic manner. The child’s jumping velocity (upwards upon leaving the trampoline) is \( v_0 \) and during the contact time \( \Delta t \) the contact force \( F(t) \) has a triangular form.

Find the necessary contact force amplitude \( F_0 \) and the jumping period \( T_0 \).

Solution  The child is subjected to its constant weight \( W = mg \) and, during the contact with the trampoline, to the contact force \( F(t) \). While the child is in the air, the equation of motion is given by

\[ m\ddot{z} = -mg. \]

Integration between \( t = t_0 = 0 \) (end of a contact) and \( t = t_1 \) (beginning of a new contact) yields

\[ mv_1 - mv_0 = -mg t_1, \]

where \( v_1 = v(t_1) \) and \( v_0 = v(0) \). Recall that the velocity at the beginning of a vertical motion and the velocity at the end of a free fall of a body are equal in magnitude: \( v_1 = -v_0 \) (note the different signs). Therefore, we obtain

\[ 2v_0 = gt_1 \quad \Rightarrow \quad t_1 = \frac{2v_0}{g}. \]
The jumping period \( T_0 \) follows with \( \Delta t = t_2 - t_1 \) as

\[
T_0 = t_1 + \Delta t \quad \rightarrow \quad T_0 = \frac{2v_0}{g} + \Delta t .
\]

We now apply the Impulse Law between time \( t_0 \) and time \( t_2 \) (see the figure):

\[
\uparrow : \quad mv_2 - mv_0 = -mgT_0 + \frac{1}{2}F_0 (t_2 - t_1) .
\]

In order to have a periodic process, the velocities \( v_2 \) and \( v_0 \) have to coincide: \( v_2 = v_0 \). With this condition the Impulse Law yields

\[
- mgT_0 + \frac{1}{2}F_0 \Delta t = 0 \quad \rightarrow \quad F_0 = \frac{2mg}{\Delta t} = 2\left(\frac{2v_0}{g\Delta t} + 1\right)m .
\]
Example 1.21 A car is travelling in a circular arc with radius $R$ and velocity $v_0$ when it starts to brake.

If the tangential deceleration is $a_t(v) = -(a_0 + \kappa v)$, where $a_0$ and $\kappa$ are given constants, find the time to brake $t_B$, the stopping distance $s_B$, and the normal acceleration $a_n$ during the braking.

Solution The acceleration $a_t$ is given as a function of the velocity: 

$$a_t(v) = \dot{v} = -(a_0 + \kappa v).$$

We separate the variables and integrate (initial condition: $v(0) = v_0$):

$$t(v) = -\int_{v_0}^{v} \frac{dv}{a_0 + \kappa v} = \frac{1}{\kappa} \ln \left( \frac{a_0 + \kappa v_0}{a_0 + \kappa v} \right).$$

The time $t_B$ when the car comes to a stop follows from the condition $v = 0$:

$$t_B = t(v = 0) = \frac{1}{\kappa} \ln \left( 1 + \frac{\kappa v_0}{a_0} \right).$$

Now we determine the inverse function of $t(v)$:

$$e^{\kappa t} = \frac{a_0 + \kappa v_0}{a_0 + \kappa v} \quad \rightarrow \quad v(t) = \frac{a_0}{\kappa} \left[ (1 + \frac{\kappa v_0}{a_0}) e^{\kappa t} - 1 \right].$$

Integration with the initial condition $s(0) = 0$ leads to the position

$$s(t) = \int_{0}^{t} v(t) dt = \frac{a_0}{\kappa^2} \left[ (1 + \frac{\kappa v_0}{a_0}) \left( 1 - e^{-\kappa t} \right) - \kappa t \right].$$

The stopping distance $s_B$ is obtained for $t = t_B$:

$$s_B = s(t_B) = \frac{a_0}{\kappa^2} \left[ (1 + \frac{\kappa v_0}{a_0}) \left( 1 - \frac{1}{1 + \frac{\kappa v_0}{a_0}} \right) - \ln \left( 1 + \frac{\kappa v_0}{a_0} \right) \right] = \frac{a_0}{\kappa^2} \left[ \frac{\kappa v_0}{a_0} \ln \left( 1 + \frac{\kappa v_0}{a_0} \right) \right].$$
The normal acceleration $a_n(t)$ during the breaking is found as

$$a_n = \frac{v^2}{R} = \frac{a_0^2}{R \kappa^2} \left\{ \left( 1 + \frac{\kappa v_0}{a_0} \right) e^{-\kappa t} - 1 \right\}^2.$$

As a check on the correctness of the result we calculate $a_n$ for $t = t_B$ and obtain $a_n = 0$. 
Example 1.22 A point $P$ moves on a parabola $y = b(x/a)^2$ from $A$ to $B$. Its position as a function of the time is given by the angle $\varphi(t) = \arctan \omega_0 t$ (see Fig. 1.47).

Determine the magnitude of velocity $v(t)$ of point $P$. How much time elapses until $P$ reaches point $B$? Calculate its velocity at $B$.

Solution The parabola $y = b(x/a)^2$ and the angle $\varphi(t) = \arctan \omega_0 t$ are given. The angle $\varphi$ can also be expressed as $\varphi = \arctan y/x$. Thus, $y/x = \omega_0 t$. Solving for $x$ and $y$ yields the position of point $P$:

$$x(t) = \frac{a^2}{b} \omega_0 t, \quad y(t) = \frac{a^2}{b} (\omega_0 t)^2.$$  

To obtain the velocity of $P$, we differentiate:

$$\dot{x}(t) = \frac{a^2}{b} \omega_0, \quad \dot{y}(t) = 2 \frac{a^2}{b} \omega_0^2 t.$$  

Thus,

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} \quad \rightarrow \quad v(t) = \frac{a^2}{b} \omega_0 \sqrt{1 + 4 \omega_0^2 t^2}.$$  

Point $B$ is reached at time $t_B$ when $y = a$:

$$x(t_B) = a \quad \rightarrow \quad t_B = \frac{b}{a \omega_0}.$$  

Thus, the velocity at $B$ is obtained as

$$v_B = v(t_B) \quad \rightarrow \quad v_B = \frac{a^2}{b} \omega_0 \sqrt{1 + 4 \frac{b^2}{a^2}}.$$
E1.23 Example 1.23 A rod with length $l$ rotates about support $A$ with angular position given by $\varphi(t) = \kappa t^2$. A body $G$ slides along the rod with position $r(t) = l(1 - \kappa t^2)$.

a) Find the magnitude of velocity and acceleration of $G$ when $\varphi = 45^\circ$.

b) At what angle $\varphi$ does $G$ hit the support?

Given: $l = 2$ m, $\kappa = 0.2$ s$^{-2}$.

Solution a) First we calculate the time $t = t_1$ that it takes the rod to reach the position $\varphi = \varphi_1 = 45^\circ$:

\[
\varphi_1 = \pi/4 = \kappa t_1^2 \quad \rightarrow \quad t_1 = \sqrt{\frac{\pi}{4\kappa}} = 1.98 \text{ s}.
\]

Then we determine the derivatives of the given functions $r(t)$ and $\varphi(t)$:

\[
r = l(1 - \kappa t^2) , \quad \dot{r} = -2\kappa lt , \quad \ddot{r} = -2\kappa l , \\
\varphi = \kappa t^2 , \quad \dot{\varphi} = 2\kappa t , \quad \ddot{\varphi} = 2\kappa.
\]

With $t = t_1$ we obtain the velocity:

\[
v_r = \dot{r} = -2\kappa lt_1 = -1.58 \text{ m/s} , \\
v_\varphi = \dot{r} \dot{\varphi} = l(1 - \kappa t_1^2)2\kappa t_1 = 0.34 \text{ m/s} , \\
v = \sqrt{v_r^2 + v_\varphi^2} = 1.62 \text{ m/s}.
\]
and the acceleration
\[ a_r = \ddot{r} - r\dot{\phi}^2 = -2kl - l(1 - \kappa t_1^2)4\kappa^2 t_1^2 \]
\[ = -1.07 \text{ m/s}^2, \]
\[ a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi} = l(1 - \kappa t_1^2)2\kappa - 2\kappa t_1 l_1 2\kappa t_1 \]
\[ = -2.34 \text{ m/s}^2, \]
\[ a = \sqrt{a_r^2 + a_\phi^2} = 2.57 \text{ m/s}^2. \]

b) Body G reaches the support \((r = 0)\) at time \(t_E\):
\[ r = 0 = l(1 - \kappa t_E^2) \quad \rightarrow \quad t_E = \sqrt{1/\kappa} = 2.24 \text{ s}. \]

This yields the corresponding angle \(\phi_E\):
\[ \phi_E = \kappa t_E^2 = \frac{1}{\kappa} \quad (\approx 57.3^\circ). \]
Example 1.24 A mouse sits in a tower (with radius $R$) at point $A$ and a cat sits at the center $0$.

If the mouse runs at a constant velocity $v_M$ along the tower wall and the cat chases it in an Archimedian spiral $r(\varphi) = R\varphi/\pi$, what must the cat’s constant velocity $v_C$ be in order to catch the mouse just as the mouse reaches its escape hole $H$? At what time does it catch the mouse?

**Solution** We determine the components of the velocity of the cat from the given equation $r(\varphi) = R\varphi/\pi$ of the Archimedian spiral:

$$v_r = \dot{r} = \frac{dr}{d\varphi} \frac{d\varphi}{dt} = \frac{R}{\pi} \dot{\varphi} \quad \text{and} \quad v_\varphi = r \dot{\varphi} = \frac{R}{\pi} \varphi \dot{\varphi}.$$ 

Thus, the constant velocity $v_C$ can be written as

$$v_C = \sqrt{v_r^2 + v_\varphi^2} = \frac{R}{\pi} \dot{\varphi} \sqrt{1 + \varphi^2} = \frac{R}{\pi} \frac{d\varphi}{dt} \sqrt{1 + \varphi^2}.$$

Separation of variables and integration lead to

$$v_C \int_0^t dt = v_C t = \frac{R}{\pi} \int_0^\varphi \sqrt{1 + \varphi^2} d\varphi.$$

With the integral

$$\int \sqrt{1 + x^2} dx = \frac{x}{2} [x \sqrt{1 + x^2} + \text{arcsinh } x]$$

this results in

$$v_C t = \frac{R}{2\pi} \left[ \varphi \sqrt{1 + \varphi^2} + \text{arcsinh } \varphi \right].$$
The time $T$ when both the cat and the mouse reach the hole follows from the constant velocity $v_M$ of the mouse along the wall:

$$\pi R = v_M T \quad \Rightarrow \quad T = \frac{\pi R}{v_M}.$$ 

Introduction of $T$ and $\varphi(T) = \pi$ into the expression for $v_C$ yields the velocity of the cat:

$$v_C = \frac{v_M}{2 \pi^2} \left( \pi \sqrt{1 + \pi^2 + \text{arcsinh} \pi} \right) = 0.62 \, v_M.$$
Example 1.25 A soccer player kicks the ball (mass $m$) so that it leaves the ground at an angle $\alpha_0$ with an initial velocity $v_0$ (Fig. 1.50). The air exerts a drag force $F_d = kv$ on the ball; it acts in the direction opposite to the velocity.

Determine the velocity $v(t)$ of the ball. Calculate the horizontal component $v_H$ of $v$ when the ball reaches the teammate at a distance $l$.

\[ x \quad z \quad v_0 \quad \alpha_0 \]

Solution The equations of motion are

\[ m\ddot{x} = -F_d \cos \alpha, \quad m\ddot{z} = mg + F_d \sin \alpha. \]

We introduce the kinematic relations

\[ \dot{x} = \frac{dx}{dt}, \quad \dot{z} = \frac{dz}{dt} \]

and the drag force $F_d = kv$. Noting that the components of the velocity are given by

\[ \dot{x} = v \cos \alpha, \quad \dot{z} = -v \sin \alpha \]

we obtain

\[ m \frac{d\dot{x}}{dt} = -k\dot{x}, \quad m \frac{d\dot{z}}{dt} = mg - k\dot{z}. \]

Separation of variables gives

\[ \frac{d\dot{x}}{\dot{x}} = -\frac{k}{m} dt, \quad \frac{d\dot{z}}{-\dot{z}} = \frac{mg}{k} dt \]

Gross, Hauger, Schröder, Wall, Govindjee
Engineering Mechanics 3, Dynamics
Springer 2013
and integration yields
\[
\ln \frac{\dot{x}}{C_1} = -\frac{k}{m} t, \quad \ln \frac{\dot{z} - mg}{k} = -\frac{k}{m} t.
\]

The constants of integration \(C_1\) and \(C_2\) can be determined from the initial conditions. For \(t = 0\) the left-hand sides of the equations above have to be zero. Thus, the arguments of the ln-functions have to be equal to one (numerator and denominator have to be equal):
\[
C_1 = \dot{x}(0) = v_0 \cos \alpha_0, \quad C_2 = \dot{z}(0) - \frac{mg}{k} = -v_0 \sin \alpha_0 - \frac{mg}{k}.
\]

This leads to the components of the velocity:
\[
\dot{x}(t) = v_0 \cos \alpha_0 e^{-kt/m}, \quad \dot{z}(t) = \frac{mg}{k} \left( \frac{mg}{k} + v_0 \sin \alpha_0 \right) e^{-kt/m}.
\]

We now integrate the \(x\)-component to obtain the horizontal position of the ball:
\[
x(t) = -\frac{m}{k} v_0 \cos \alpha_0 e^{-kt/m} + C.
\]

Exploiting the initial condition \(x(0) = 0\) → \(C = \frac{m}{k} v_0 \cos \alpha_0\) yields
\[
x(t) = -\frac{m}{k} v_0 \cos \alpha_0 \left(1 - e^{-kt/m}\right).
\]

The time \(t^*\) when the ball reaches the teammate at the distance \(l\) is found to be
\[
x(t^*) = l \Rightarrow t^* = \frac{m}{k} \ln \frac{mv_0 \cos \alpha_0}{mv_0 \cos \alpha_0 - kl}.
\]

This leads to the corresponding horizontal component of the velocity:
\[
\dot{x}(t^*) = v_0 \cos \alpha_0 \frac{kl}{m}.
\]
Example 1.26 A body with mass \( m \) starts at a height \( h \) at time \( t = 0 \) with an initial horizontal velocity \( v_0 \).

If the wind resistance can be approximated by a horizontal force \( H = c_0 \, m \, \dot{x}^2 \), at what time \( t_B \) and location \( x_B \) does it hit the ground?

Solution The equations of motion are given by

\[ \rightarrow: \quad m \ddot{x} = -mc_0 \dot{x}^2, \quad \uparrow: \quad m \ddot{z} = -mg. \]

We integrate and apply the initial conditions \( x(0) = 0, z(0) = h, \dot{x}(0) = v_0, \dot{z}(0) = 0 \) to obtain

\[
\int_{0}^{\ddot{x}} \frac{1}{\dot{x}} \, dt = -c_0 \int_{0}^{t} \, dt \quad \rightarrow \quad \dot{x} = \frac{1}{v_0} + c_0 t, \\
\int_{0}^{\ddot{x}} \, dt = \int_{0}^{t} \frac{1}{v_0 + c_0 t} \, dt \quad x = \frac{1}{c_0} \ln(1 + c_0 v_0 \sqrt{2h/g}), \\
\dot{z} = -gt, \quad z = -\frac{g}{2}t^2 + h.
\]

The time \( t_B \) when the body hits the ground follows from \( z_B = 0 \):

\[ t_B = \sqrt{\frac{2h}{g}}. \]

Thus, the location of point \( B \) is obtained as

\[ x_B = x(t = t_B) = \frac{1}{c_0} \ln \left( 1 + c_0 v_0 \sqrt{\frac{2h}{g}} \right). \]

In the case of a vanishing wind resistance \( (c_0 = 0) \) this result reduces to

\[ x_B = \lim_{c_0 \to 0} \frac{1}{c_0} \ln \left( 1 + c_0 v_0 \sqrt{\frac{2h}{g}} \right) = v_0 \sqrt{\frac{2h}{g}}. \]
Example 1.27 A mass $m$ slides on a rotating frictionless and massless rod $S$ such that it is pressed against a rough circular wall (coefficient of friction $\mu$).

If the mass starts in contact with the wall at a velocity $v_0$, how many rotations will it take for its velocity to drop to $v_0/10$?

Solution If we express the acceleration vector in terms of the Serret-Frenet frame, then the equations of motion are written as

\[
\begin{aligned}
\hat{v} : m\ddot{v} &= -R, \\
\hat{n} : m\ddot{n} &= N.
\end{aligned}
\]

Note that the weight of the mass does not influence the motion since the motion takes place in a horizontal plane. If we use the kinematic relations $\ddot{v} = \dot{v}$, $\ddot{n} = v^2/r$, and the friction law $R = \mu N$ we can write the first equation of motion in the form

\[
m\dot{v} = -\mu m \frac{v^2}{r}.
\]

Separation of variables and integration yield

\[
\int_{v_0}^{v} \frac{dv}{v^2} = -\int_{0}^{t} \mu \frac{dt}{\dot{v}} \Rightarrow v(t) = \frac{v_0}{1 + \frac{\mu v_0}{r} t}.
\]

We integrate again to obtain the distance traveled:

\[
\int_{0}^{s} ds = v_0 \int_{0}^{t} \frac{\frac{1}{1 + \frac{\mu v_0}{r} t}}{1 + \frac{\mu v_0}{r} t} dt \Rightarrow s(t) = \frac{r}{\mu} \ln \left(1 + \frac{\mu v_0}{r} t \right).
\]
Now we calculate the time $t_1$ that it takes for the velocity to drop to $v_0/10$:

$$\frac{v_0}{10} = \frac{v_0}{1 + \frac{\mu v_0}{r} t_1} \quad \Rightarrow \quad t_1 = \frac{9r}{\mu v_0}.$$

The corresponding value $s(t_1)$ is found as

$$s_1 = s(t_1) = \frac{r}{\mu} \ln \left(1 + \frac{\mu v_0}{r} t_1\right) = \frac{r}{\mu} \ln 10.$$

This yields the corresponding number of rotations:

$$n = \frac{s_1}{2\pi r} = \frac{\ln 10}{2\pi \mu}.$$
Example 1.28 A car (mass $m$) is traveling with constant velocity $v$ along a banked circular curve (radius $r$, angle of slope $\alpha$), see Fig. 1.53. The coefficient of static friction $\mu_0$ between the tires of the car and the surface of the road is given.

Determine the region of the allowable velocities so that sliding (down or up the slope) does not take place.

Solution We model the car as a point mass and express the acceleration vector in terms of the Serret-Frenet frame. Then the equation of motion in the direction of the normal vector is (see the free-body diagram)

\[ ma_n = N \sin \alpha + H \cos \alpha. \]

Since the car does not move in the vertical direction we can apply the equilibrium condition

\[ 0 = N \cos \alpha - H \sin \alpha - mg. \]
We now solve these two equations for the normal force \( N \) and the force of static friction \( H \):

\[
H = ma_n \cos \alpha - mg \sin \alpha ,
\]

\[
N = ma_n \sin \alpha + mg \cos \alpha .
\]

The car does not slide if the condition of static friction

\[
|H| \leq \mu_0 N
\]

is satisfied. With \( a_n = \frac{v^2}{r} \) this leads to the allowable region of the velocity:

\[
\frac{\tan \alpha - \mu_0}{1 + \mu_0 \tan \alpha} \leq \frac{v^2}{gr} \leq \frac{\tan \alpha + \mu_0}{1 - \mu_0 \tan \alpha}.
\]

This is displayed for \( \mu = 0.3 \) as a function of the angle \( \alpha \) in the figure.
**Example 1.29** A car (mass $m$) has a velocity $v_0$ at the beginning of a curve (Fig. 1.54). Then it slows down with constant tangential acceleration $a_t = -a_0$. The coefficient of static friction between the road and the tires is $\mu_0$.

Calculate the velocity $v$ of the car as a function of the arc-length $s$. What is the necessary radius of curvature $\rho(s)$ of the road so that the car does not slide?

**Solution** We model the car as a point mass. Its velocity $v$ can be determined from the given constant tangential acceleration $a_t$ through integration (initial condition $v(0) = v_0$):

$$\dot{v} = a_t = -a_0 \quad \rightarrow \quad v(t) = v_0 - a_0 t.$$  

The distance traveled follows from:

$$\dot{s} = v \quad \rightarrow \quad s(t) = s_0 + v_0 t - \frac{a_0 t^2}{2}.$$  

With the initial condition $s(0) = s_0 = 0$ we obtain

$$s(t) = v_0 t - \frac{a_0 t^2}{2}.$$  

Elimination of the time $t$ yields the velocity as a function of the arc-length:

$$v(s) = \sqrt{v_0^2 - 2a_0 s}.$$
In order to determine the necessary radius of curvature \( \rho(s) \) we write down the equations of motion:

\[
ma_t = H_t \quad \rightarrow \quad H_t = -ma_0 ,
ma_n = H_n \quad \rightarrow \quad H_n = m \frac{v^2}{\rho} ,
0 = N - mg \quad \rightarrow \quad N = mg .
\]

The static friction force \( H = (H_t^2 + H_n^2)^{1/2} \) and the normal force \( N \) have to satisfy the condition of static friction to avoid sliding of the car:

\[
|H| \leq \mu_0 N \quad \rightarrow \quad a_0^2 + \frac{v^4}{\rho^2} \leq \mu_0^2 g^2 .
\]

Solving for \( \rho \) yields

\[
\rho(s) \geq \frac{v_0^2 - 2a_0s}{\sqrt{\mu_0^2 g^2 - a_0^2}} .
\]

This condition can not be satisfied if \( a_0 > \mu_0 g \).
Example 1.30  A bowling ball (mass \(m\)) moves with constant velocity \(v_0\) on the frictionless return of a bowling alley. It is lifted on a circular path (radius \(r\)) to the height \(2r\) at the end of the return. The upper part of the circular path has a frictionless guide of length \(r\varphi_G\) (Fig. 1.55).

Given the angle \(\varphi_G\), determine the velocity \(v_0\) such that the bowling ball reaches the upper level.

Solution  The velocity \(v_0\) has to be large enough so that the bowling ball reaches the upper level (height \(2r\)) with a velocity \(v \geq 0\). It is convenient to use the Conservation of Energy Law to determine the relation between \(v_0\) and \(v\). With \(T_0 = \frac{mv_0^2}{2}, V_0 = 0, T_1 = \frac{mv^2}{2}\) and \(V_1 = 2mgr\) we obtain

\[T_0 + V_0 = T_1 + V_1 \rightarrow \frac{mv_0^2}{2} + 0 = \frac{mv^2}{2} + 2mgr.\]

This yields a first requirement on \(v_0\):

\[v_0^2 = v^2 + 4gr \rightarrow v_0^2 \geq 4gr.\]

In addition, the velocity \(v_0\) has to be large enough so that the normal force between the bowling ball and the lower part of the circular path does not become zero before the ball reaches the guide of length \(r\varphi_G\). The necessary velocity \(v(\varphi_G)\) follows from the equation of motion in the normal direction (note that \(\varphi\) is measured from the upper level):

\[m \frac{d^2 \varphi}{dt^2} = N - mg \cos \varphi.\]
The condition $N \geq 0$ for $\varphi = \varphi_G$ leads to

$$v^2(\varphi_G) \geq gr \cos \varphi_G.$$ 

In order to calculate the corresponding velocity $v_0$ we use the Conservation of Energy Law between the lower level and the beginning of the guide:

$$T_0 + V_0 = T_2 + V_2$$

$$\rightarrow \quad mv_0^2/2 + 0 = mv^2(\varphi_G)/2 + mgr(1 + \cos \varphi_G).$$

Thus, we obtain the second requirement

$$v_0^2 \geq (2 + 3 \cos \varphi_G)gr.$$ 

Both requirements are plotted in the figure, which shows that the minimum velocity $v_0$ is obtained as

$$v_0^2 = \begin{cases} 
(2 + 3 \cos \varphi_G)gr & \text{for } \varphi_G < \varphi^*_G = \arccos 2/3, \\
4gr & \text{for } \varphi_G > \varphi^*_G.
\end{cases}$$
Example 1.31 A point mass \( m \) is subjected to a central force \( F = mk^2r \), where \( k \) is a constant and \( r \) is the distance of the mass from the origin 0. At time \( t = 0 \) the mass is located at \( P_0 \) and has velocity components \( v_x = v_0 \) and \( v_y = 0 \).

Find the trajectory of the mass.

Solution The equations of motion in the directions \( x \) and \( y \) are

\[
\begin{align*}
\rightarrow: \quad m\ddot{x} &= -F \cos \alpha, \\
\uparrow: \quad m\ddot{y} &= -F \sin \alpha.
\end{align*}
\]

With \( F = mk^2r \), \( x = r \cos \alpha \) and \( y = r \sin \alpha \) we obtain

\[
\ddot{x} + k^2x = 0, \quad \ddot{y} + k^2y = 0.
\]

Both equations are equivalent to the differential equation that describes undamped free vibrations of a point mass (see Chapter 5). The solutions are

\[
x = A \cos kt + B \sin kt, \quad y = C \cos kt + D \sin kt.
\]

The constants of integration are calculated from the initial conditions:

\[
\begin{align*}
x(0) &= 0 \quad \rightarrow \quad A = 0, \\
y(0) &= r_0 \quad \rightarrow \quad C = r_0, \\
\dot{x}(0) &= v_0 \quad \rightarrow \quad B = \frac{v_0}{k}, \\
\dot{y}(0) &= 0 \quad \rightarrow \quad D = 0.
\end{align*}
\]
We now have a parametric representation of the curve describing the motion:

\[ x = \frac{v_0}{k} \sin kt , \quad y = r_0 \cos kt . \]

To eliminate the time, these equations are squared and then added. Thus, we finally obtain

\[ \left( \frac{x}{v_0/k} \right)^2 + \left( \frac{y}{r_0} \right)^2 = 1 . \]

The point mass moves along an ellipse which reduces to a circle for \( k = v_0/r_0 \).
Example 1.32  A centrifuge with radius \( r \) rotates with constant angular velocity \( \omega_0 \). A mass \( m \) is to be placed at rest in the centrifuge and accelerated within a time \( t_1 \) to the angular velocity \( \omega_0 \).

What will be the needed (constant) moment \( M \) acting on the mass? What is the power \( P \) of this moment?

Solution  The centrifuge rotates with a constant angular velocity. Thus, the driving torque \( M \) needed to accelerate the point mass is equal to the moment of the friction force \( R \) which acts between the mass and the cylinder:

\[
M = rR.
\]

The equation of motion in the tangential direction for the mass is

\[
\uparrow: \quad ma_t = R,
\]

where \( a_t = r\dot{\omega} \). Thus,

\[
mr\ddot{\omega} = \frac{M}{r} \quad \Rightarrow \quad \dot{\omega} = \frac{M}{mr^2} t.
\]

Integration with the initial condition \( \omega(0) = 0 \) yields

\[
\omega = \frac{M}{mr^2} t.
\]
The condition $\omega(t_1) = \omega_0$ leads to the required moment:

$$M = \frac{r^2 m\omega_0}{t_1}.$$ 

The corresponding power is

$$P = M \cdot \omega_0 = M\omega_0 = \frac{r^2 m\omega_0^2}{t_1}.$$ 

Note that the moment $M$ has to be reduced to zero after time $t_1$. Otherwise the mass would be further accelerated.
Example 1.33  A skier (mass $m$) has the velocity $v_A = v_0$ at point $A$ of the cross country course (Fig. 1.58). Although he tries hard not to lose velocity skiing uphill, he reaches point $B$ with only a velocity $v_B = 2v_0/5$. Skiing downhill between point $B$ and the finish $C$ he again gains speed and reaches $C$ with $v_C = 4v_0$. Between $B$ and $C$ assume that a constant friction force acts due to the soft snow in this region; the drag force from the air on the skier can be neglected.

Calculate the work done by the skier on the path from $A$ to $B$ (here the friction force is negligible). Determine the coefficient of kinetic friction between $B$ and $C$.

Solution  In order to calculate the work done by the skier on the path from $A$ to $B$ we use the work-energy theorem:

$$T_B + V_B = T_A + V_A + U$$

$$\rightarrow \frac{1}{2}mv_B^2 + mgh = \frac{1}{2}mv_A^2 + 0 + U$$

$$\rightarrow U = mgh - \frac{21}{50} \frac{1}{2}v_0^2.$$
Now we apply the work-energy theorem between points $B$ and $C$, where the work of the friction force is given by $-R_{BC}$:

$$T_C + V_C = T_B + V_B + U$$

$$\rightarrow \quad mv_C^2/2 + 0 = mv_B^2/2 + 3mgh - R_{BC}.$$  

With the normal force

$$N = mg \cos \alpha \rightarrow N = mg \frac{10h}{l_{BC}}$$

and the law of kinetic friction

$$R = \mu N \rightarrow R = \mu mg \frac{10h}{l_{BC}}$$

we obtain

$$\mu = \frac{3}{10} \frac{v_C^2 - v_B^2}{20gh} \rightarrow \mu \approx \frac{3}{10} \frac{4v_B^2}{5gh}.$$  

The result is valid only for $\mu \geq 0$.  

Gross, Hauger, Schröder, Wall, Govdijee
Engineering Mechanics 3, Dynamics
Springer 2013
**Example 1.34** A circular disk (radius $R$) rotates with the constant angular velocity $\Omega$. A point $P$ moves along a straight guide; its distance from the center of the disk is given by $\xi = R \sin \omega t$ where $\omega = \text{const}$ (Fig. 1.59).

Determine the velocity and the acceleration of $P$.

**Solution** We use polar coordinates $r, \varphi$ to solve the problem. From $\Omega = \text{const}$ we find

$$\dot{\varphi} = \Omega = \text{const} \rightarrow \varphi = \Omega t, \quad \ddot{\varphi} = 0.$$  

The position vector is written as

$$\mathbf{r} = \xi \mathbf{e}_r \rightarrow \mathbf{r} = R \sin \omega t \mathbf{e}_r.$$  

Differentiation yields the velocity vector

$$\mathbf{v} = \dot{\xi} \mathbf{e}_r + \xi \dot{\varphi} \mathbf{e}_\varphi \rightarrow \mathbf{v} = R \omega \cos \omega t \mathbf{e}_r + R \Omega \sin \omega t \mathbf{e}_\varphi$$

and the acceleration vector

$$\mathbf{a} = (\ddot{\xi} - \xi \ddot{\varphi}^2) \mathbf{e}_r + (\dot{\xi} \ddot{\varphi} + 2 \dot{\xi} \dot{\varphi}) \mathbf{e}_\varphi$$

$$\rightarrow \mathbf{a} = -R (\omega^2 + \Omega^2) \sin \omega t \mathbf{e}_r + 2 R \omega \Omega \cos \omega t \mathbf{e}_\varphi.$$
The paths of point $P$ are displayed for several values of $\Omega/\omega$ in the following figures.

- $\Omega/\omega = 0.25$
- $\Omega/\omega = 0.5$
- $\Omega/\omega = 1.0$
- $\Omega/\omega = 2.0$
Example 1.35  A chain with length \( l \) and mass \( m \) hangs over the edge of a frictionless table by an amount \( e \).

If the chain starts with zero initial velocity, find the position of the end of the chain as a function of time.

Solution  All the links of the chain have the same displacement, velocity and acceleration. The corner only produces a change of direction. We therefore consider the chain to be a single mass with an applied force that depends on the length \( x \) of the overhanging part. Thus, with \( a = \ddot{x} \), the equation of motion is

\[
ma = m\frac{x}{l}g \Rightarrow \ddot{x} - \frac{g}{l}x = 0.
\]

This differential equation of second order with constant coefficients has the solution

\[
x(t) = A \cosh \sqrt{\frac{g}{l}} t + B \sinh \sqrt{\frac{g}{l}} t.
\]

We calculate the integration constants from the initial conditions:

\[
\begin{align*}
\dot{x}(0) = 0 & \Rightarrow B = 0 \\
x(0) = e & \Rightarrow A = e
\end{align*}
\]

This solution is valid only for \( x \leq l \).

We may also solve the problem by making an imaginary section cut at the corner. Then the equations of motion for each part of the chain are

\[
\frac{d}{dt} \left( \frac{ml - x}{l} \right) = S, \quad \frac{d}{dt} \left( \frac{m}{l} \ddot{x} \right) = \frac{m}{l} x g - S.
\]

If we eliminate \( S \) we again obtain the differential equation

\[
\ddot{x} - \frac{g}{l}x = 0.
\]
Example 1.36 A rotating source of light throws a beam of light onto a screen (Fig. 1.61). Point \( P \) on the screen should move with constant velocity \( v_0 \).

Determine the required angular acceleration \( \ddot{\varphi}(\varphi) \) of the source of light. Sketch \( \dot{\varphi}(\varphi) \) and \( \ddot{\varphi}(\varphi) \).

Solution The position of point \( P \) is given by

\[
x = r_0 \tan \varphi.
\]

We write down the inverse function

\[
\varphi = \arctan \frac{x}{r_0}.
\]

Differentiation with \( \dot{x} = v = v_0 = \text{const} \) yields

\[
\dot{\varphi} = \frac{1}{1 + \left( \frac{x}{r_0} \right)^2} \frac{\dot{x}}{r_0} = \frac{v_0 r_0}{r_0^2 + x^2} \frac{v_0}{r_0 (1 + \tan^2 \varphi)} = \frac{v_0}{r_0} \cos^2 \varphi,
\]

\[
\ddot{\varphi} = \frac{v_0}{r_0} 2 \cos \varphi (- \sin \varphi) \dot{\varphi} = 2 \left( \frac{v_0}{r_0} \right)^2 \sin \varphi \cos \varphi.
\]
Note: the maximum value of $\ddot{\varphi}$, located at $\varphi = \pm 30^\circ$, is obtained as

$$|\ddot{\varphi}_{\text{max}}| = \frac{3}{8} \sqrt{3} \left(\frac{v_0}{r_0}\right)^2.$$
Example 1.37  A car travels with velocity $v_0 = 100$ km/h. At time $t = 0$, the driver fully applies the brakes. At that moment the car starts sliding on a rough road (coefficient of kinetic friction $\mu$).

Use the simplest model for the car (point mass) and calculate the time $t^*$ and the distance $x^*$ until the car comes to a stop

(a) on a dry road ($\mu = 0.8$),

(b) on a wet road ($\mu = 0.35$).

Solution The equation of motion in the horizontal direction

$\leftarrow: \quad m \ddot{x} = -R,$

the equilibrium condition in the vertical direction

$\uparrow: \quad 0 = N - mg$

and the law of friction

$R = \mu N$

lead to

$\ddot{x} = -\mu g.$

Integration with $v(t=0) = v_0$ and $x(t=0) = 0$ yields

$v(t) = v_0 - \mu gt, \quad x(t) = v_0 t - \frac{1}{2} \mu gt^2.$

The time $t^*$ follows from the condition $v = 0$:

$t^* = \frac{v_0}{\mu g}.$

This leads to

$x^* = x(t^*) = \frac{v_0^2}{\mu g}, \quad \frac{v_0^2}{2 \mu g} = \frac{v_0^2}{2 \mu g}.$
a) On a dry road ($\mu = 0.8$) we obtain
\[
    t^* = \frac{100 \cdot 1000}{3600 \cdot 0.8 \cdot 9.81} = 3.55 \text{ s},
\]
\[
    x^* = \left( \frac{100}{3.6} \right)^2 \frac{1}{2 \cdot 0.8 \cdot 9.81} = 49 \text{ m}.
\]

b) On a wet road ($\mu = 0.35$) we are led to
\[
    t^* = \frac{100 \cdot 1000}{3600 \cdot 0.35 \cdot 9.81} = 8.1 \text{ s},
\]
\[
    x^* = \left( \frac{100}{3.6} \right)^2 \frac{1}{2 \cdot 0.35 \cdot 9.81} = 112 \text{ m}.
\]
**Example 1.38** In order to determine the coefficient of restitution $e$ experimentally, a ball is dropped from height $h_0$ onto a horizontal rigid surface (Fig. 1.62). After the ball has hit the surface 7 times, it reaches only 20% of the original height $h_0$.

Calculate the coefficient of restitution.

**Solution** With the positive direction of the velocity taken as upwards, the velocity of the ball immediately before the first impact is given by

$$v = -\sqrt{2gh_0} .$$

The definition

$$e = -\frac{\overline{v}}{v}$$

of the coefficient of restitution in terms of the velocities yields the velocity $\overline{v}$ immediately after the impact:

$$\overline{v} = -ev = e\sqrt{2gh_0} .$$

From the Conservation of Energy Law, we obtain the height which the body will reach:

$$\frac{1}{2} mv^2 = mgh_1 \quad \rightarrow \quad h_1 = \frac{v^2}{2g} = e^2 h_0 .$$

Similarly, we obtain

$$h_i = e^2 h_{i-1}$$
for the subsequent impacts. Thus,

\[ h_7 = e^2 h_6 = e^4 h_5 = \ldots = e^{14} h_0 \]

and

\[ e = \left( \frac{h_7}{h_0} \right)^{1/14} = 0.2^{1/14} \approx 0.891. \]
Example 2.10 Two vehicles (masses $m_1$ and $m_2$, velocities $v_1$ and $v_2$) crash head-on, see Fig. 2.20. After a plastic impact the vehicles are entangled and slide with locked wheels a distance $s$ to the right.

The coefficient of kinetic friction between the wheels and the road is $\mu$.

Calculate $v_1$ if $v_2$ and $s$ are known.

Solution The total momentum of the system remains unchanged during the collision. We assume that positive velocities are directed to the right (note the direction of $v_2$). We then obtain the velocity $\bar{v}$ of the vehicles after the impact:

$$m_1v_1 - m_2v_2 = (m_1 + m_2)\bar{v} \quad \Rightarrow \quad \bar{v} = \frac{m_1v_1 - m_2v_2}{m_1 + m_2}.$$

In order to relate the distance $s$ to the velocity $\bar{v}$ we apply the work-energy theorem

$$T_1 - T_0 = U.$$

We insert the kinetic energies

$$T_0 = (m_1 + m_2)\bar{v}^2/2, \quad T_1 = 0,$$

the work of the friction force

$$U = -Rs \quad \Rightarrow \quad R = \mu N \quad \Rightarrow \quad R = \mu(m_1 + m_2)g,$$

and Coulomb’s friction law

$$R = \mu N \quad \Rightarrow \quad R = \mu(m_1 + m_2)g.$$ 

We then obtain

$$-\frac{1}{2}(m_1 + m_2)\bar{v}^2 = -\mu(m_1 + m_2)gs$$

and

$$v_1 = \frac{m_2}{m_1}v_2 + \left(1 + \frac{m_2}{m_1}\right)\sqrt{2\mu gs}.$$
Example 2.11 A block (mass \(m_2\)) rests on a horizontal platform (mass \(m_1\)) which is also initially at rest (Fig. 2.21). A constant force \(F\) accelerates the platform (wheels rolling without friction) which causes the block to slide on the rough surface of the platform (coefficient of kinetic friction \(\mu\)).

Determine the time \(t^*\) that it takes the block to fall off the platform.

**Solution** We separate the block and the platform and introduce the coordinates \(x_1\) and \(x_2\) as shown in the free-body diagram. Then the equations of motion for the two bodies are

\[
\begin{aligned}
\text{(1) } & \quad m_1\ddot{x}_1 = F - R, \\
\text{(2) } & \quad m_2\ddot{x}_2 = R.
\end{aligned}
\]

With the friction law \(R = \mu N = \mu m_2 g\), we obtain

\[
\begin{aligned}
\ddot{x}_1 &= \frac{F - \mu m_2 g}{m_1}, \\
\ddot{x}_2 &= \mu g.
\end{aligned}
\]

Now, we integrate twice. Using the initial conditions \(x_1(0) = x_2(0) = 0\) and \(\dot{x}_1(0) = \dot{x}_2(0) = 0\) we obtain

\[
\begin{aligned}
\dot{x}_1 &= \frac{F - \mu m_2 g}{m_1} t, \\
\dot{x}_2 &= \mu g t, \\
x_1 &= \frac{F - \mu m_2 g}{2m_1} t^2, \\
x_2 &= \mu g \frac{t^2}{2}.
\end{aligned}
\]
The block falls off the platform if

\[ x_1 - x_2 = l. \]

This yields

\[ F - \mu m_1 g + \frac{1}{2} m_1 \left( (x_1')^2 - \frac{m_1}{m_1 + m_2} (l)^2 \right) = l \rightarrow t^* = \sqrt{\frac{2ml}{F - \mu m_1 g + m_2 g}}. \]
Example 2.12  A railroad wagon (mass $m_1$) has a velocity $v_1$ (Fig. 2.22). It collides with a wagon (mass $m_2$) which is initially at rest. Both wagons roll without friction after the collision. The second wagon is connected via a spring (spring constant $k$) with a block (mass $m_3$) that lies on a rough surface (coefficient of static friction $\mu_0$).

Assume the impact to be plastic and determine the maximum value of $v_1$ so that the block stays at rest.

Solution  The total momentum of the system remains unchanged during the impact. This yields the velocity $\bar{v}$ of the wagons after the collision:

$$m_1 v_1 = (m_1 + m_2) \bar{v} \quad \rightarrow \quad \bar{v} = \frac{m_1}{m_1 + m_2} v_1 .$$

We assume that the block stays at rest. Then we can write down the equilibrium conditions:

\begin{align*}
\rightarrow: & \quad H = F, \quad \uparrow: \quad N = m_3 g .
\end{align*}
The force of static friction $H$ and the normal force $N$ have to satisfy the condition of static friction:

$$H \leq \mu_0 N \quad \rightarrow \quad F \leq \mu_0 m_3 g .$$

With

$$F = kx \quad \rightarrow \quad x \leq \frac{\mu_0 m_3 g}{k}$$

we obtain the maximum compression of the spring:

$$x_{\text{max}} = \frac{\mu_0 m_3 g}{k} .$$

Now we apply the Conservation of Energy Law to determine $v_{1\text{max}}$:

$$\frac{1}{2}(m_1 + m_2)v^2 = \frac{1}{2}kx_{\text{max}}^2 \quad \rightarrow \quad v_{1\text{max}} = \frac{\mu_0 m_3 g}{m_1 + m_2} \sqrt{\frac{m_1 + m_2}{k}} .$$
**Example 2.13** A point mass $m_1$ strikes a point mass $m_2$ which is suspended from a string (length $l$, negligible mass) as shown in Fig. 2.23. The maximum force $S^*$ that the string can sustain is given.

Assume an elastic impact and determine the velocity $v_0$ that causes the string to break.

**Solution** First we formulate the conservation of linear momentum

$$m_1 v_0 = m_1 \bar{v}_1 + m_2 \bar{v}_2$$

and of energy (elastic impact!):

$$m_1 v_0^2 / 2 = m_1 \bar{v}_1^2 / 2 + m_2 \bar{v}_2^2 / 2.$$ 

From these equations we can calculate the velocity $\bar{v}_2$ of mass $m_2$ immediately after the impact:

$$\bar{v}_2 = \frac{2m_1}{m_1 + m_2} v_0.$$ 

Now, we write down the equation of motion

$$m_2 a_n = S - m_2 g \cos \varphi$$

With $a_n = \bar{v}_2^2 / l$ we obtain the force in the string:

$$S = m_2 \bar{v}_2^2 / l + m_2 g \cos \varphi.$$
The maximum force $S_{\text{max}}$ in the string is found for $\varphi = 0$:

$$S_{\text{max}} = m_2 \left( \frac{\dot{v}_2^2}{l + g} \right).$$

The string breaks if the maximum force $S_{\text{max}}$ is larger than the allowable force $S^*$:

$$S_{\text{max}} > S^*. $$

This yields

$$v_0 > \frac{m_1 + m_2}{2m_1} \sqrt{l \left( \frac{S^*}{m_2} - g \right)}. $$
Example 2.14  A ball \( \oplus \) (mass \( m_1 \)) hits a second ball \( \ominus \) (mass \( m_2 \), velocity \( v_2 = 0 \)) with a velocity \( v_1 \) as shown in Fig. 2.24. Assume that the impact is partially elastic (coefficient of restitution \( e \)) and all surfaces are smooth.

Given: \( r_2 = 3r_1 \), \( m_2 = 4m_1 \).

Determine the velocities of the balls after the collision.

Solution  We introduce the auxiliary angle \( \alpha \). Since the surfaces are smooth, the linear impulse \( \hat{F} \) acts in the direction of the line of impact and the Impulse Laws are given by

\[ \oplus \rightarrow : \quad m_1(v_1 - v_1) = -\hat{F} \cos \alpha, \]

\[ \ominus \leftarrow : \quad m_2v_2 = \hat{F}. \]

Note that the weights of the balls can be neglected during impact and that ball \( \oplus \) moves only horizontally.

With the hypothesis

\[ e = \frac{\overline{v}_{1x} - \overline{v}_{2x}}{v_{1x} - v_{2x}}, \]

and

\[ v_{1x} = v_1 \cos \alpha, \quad v_{2x} = 0, \quad \overline{v}_{1x} = \overline{v}_1 \cos \alpha, \quad \overline{v}_{2x} = \overline{v}_2 \]

we obtain

\[ \overline{v}_1 = \frac{v_1}{1 + \frac{m_2}{m_1} \cos^2 \alpha}, \quad \overline{v}_2 = \frac{v_1 (1 + e \cos \alpha)}{1 + \frac{m_2}{m_1} \cos^2 \alpha}. \]
Introduction of $m_2 = 4m_1$ and

$$\sin \alpha = \frac{r_2 - r_1}{r_1 + r_2} = \frac{1}{2} \quad \Rightarrow \quad \cos \alpha = \sqrt{1 - \sin^2 \alpha} = \frac{\sqrt{3}}{2}$$

yields

$$\bar{v}_1 = \frac{1 - 3e}{4} v_1, \quad \bar{v}_2 = \frac{\sqrt{3}}{8} (1 + e) v_1.$$
Example 2.15 A hunter (mass $m_1$) sits in a boat (mass $m_2 = 2m_1$) which can move in the water without resistance. The boat is initially at rest.

a) Determine the velocity $v_{B_1}$ of the boat after the hunter fires a bullet (mass $m_3 = m_1/1000$) with a velocity $v_0 = 500$ m/s.

b) Find the direction of the velocity of the boat after a second shot is fired at an angle of 45° with respect to the first one.

Solution  

a) We introduce a coordinate system and assume that the bullet is fired in the direction of the negative $x$-axis. Since the linear momentum before the firing of the bullet is zero, the total linear momentum of the system after the firing (velocities positive to the right) also has to be zero:

\[ \rightarrow : (m_1 + m_2 - m_3)v_{B_1} - m_3v_0 = 0, \]

This yields the velocity $v_{B_1}$ of the boat:

\[ v_{B_1} = \frac{m_3}{m_1 + m_2 - m_3} v_0 = \frac{1}{1 + 2 - \frac{1}{1000}} \cdot \frac{500}{2999} \approx 0.167 \text{ m/s}. \]

The algebraic sign shows that the boat moves in the direction of the positive $x$-axis.

b) Now, we have to formulate the Impulse Laws in the $x$- and $y$-direction:

\[ \rightarrow : (m_1 + m_2 - m_3)v_{B_1} = (m_1 + m_2 - 2m_3)v_{B_2} - m_3v_0 \cos 45°, \]

\[ \uparrow : 0 = (m_1 + m_2 - 2m_3)v_{B_2} - m_3v_0 \sin 45°. \]
They lead to

\[ v_{B_2} = \frac{m_1 + m_2 - m_3}{m_1 + m_2 - 2m_3} v_{B_1} + \frac{m_1}{m_1 + m_2 - 2m_3} v_0 \cos 45^\circ \]

\[ \approx 0.285 \text{ m/s}, \]

\[ w_{B_2} = \frac{m_3}{m_1 + m_2 - 2m_3} v_0 \sin 45^\circ \approx 0.118 \text{ m/s}, \]

\[ \rightarrow \tan \alpha = \frac{w_{B_2}}{v_{B_2}} = 0.414 \rightarrow \alpha = 22.5^\circ. \]
Example 2.16 A car ② goes into a skid on a wet road and comes to a stop sideways across the road as shown in Fig. 2.25. In spite of having fully applied the brakes a distance $s_1$ from car ②, a second car ① (sliding with the coefficient of kinetic friction $\mu$) collides with car ②. This causes car ② to slide an additional distance $s_2$. Assume a partially elastic central impact. Given: $m_1 = 2m_2$, $\mu = 1/3$, $e = 0.2$, $s_1 = 50\, \text{m}$, $s_2 = 10\, \text{m}$.

Determine the velocity $v_0$ of car ① before the brakes were applied.

Solution The velocity $v_1$ of car ① immediately before impact follows from the work-energy theorem:

$$\frac{1}{2}m_1v_1^2 - \frac{1}{2}mv_0^2 = -(\mu m_1 g)s_1 \quad \Rightarrow \quad v_1^2 = v_0^2 - 2\mu m_1 g s_1.$$ 

The velocity $v_2$ of car ① immediately after impact can be determined from the conservation of linear momentum (note $v_2 = 0$)

$$m_1 v_1 = m_1 v_1 + m_2 v_2$$

and the hypothesis

$$e = \frac{v_2 - v_1}{v_1}.$$ 

We obtain

$$v_2 = \frac{1 + e}{1 + \frac{m_2}{m_1}} v_1.$$ 

Now we apply the work-energy theorem to the sliding of car ②:

$$-\frac{1}{2}m_2 v_2^2 = -(\mu m_2 g) s_2 \quad \Rightarrow \quad v_2^2 = 2\mu m_2 g s_2.$$
If we eliminate \( v_1 \) and \( v_2 \) we obtain
\[
v_0^2 = \left( \frac{1 + \frac{m_2}{m_1}}{1 + e} \right)^2 \left( 2 \mu \, g s_2 + 2 \mu \, g s_1 \right)
\]
\[
= \left( \frac{1.5}{1.2} \right)^2 \cdot \frac{2}{3} \cdot 9.81 \cdot 10 + \frac{2}{3} \cdot 9.81 \cdot 50 = 420.19 \text{ m}^2/\text{s}^2
\]
or
\[
v_0 = 20.7 \text{ m/s} \quad \rightarrow \quad v_0 = 74.6 \text{ km/h}.
\]
Note: the velocity of car \( \Theta \) at impact is \( v_1 = 36.4 \text{ km/h} \).
Example 2.17 Two cars (point masses $m_1$ and $m_2$) collide at an intersection with velocities $v_1$ and $v_2$ at an angle $\alpha$ (Fig. 2.26). Assume a perfectly plastic collision.

Determine the magnitude and the direction of the velocity immediately after the impact. Calculate the loss of energy during the collision.

Solution Both cars have the same velocity $\mathbf{v}$ after the collision (plastic impact). We write down the Impulse Laws in the $x$- and $y$-directions:

\[ \rightarrow: \quad m_1 v_1 + m_2 v_2 \cos \alpha = (m_1 + m_2) v \cos \beta, \]
\[ \uparrow: \quad m_2 v_2 \sin \alpha = (m_1 + m_2) v \sin \beta. \]

If we square and then add these equations we obtain

\[ \sqrt{\mathbf{v}^2} = \frac{1}{m_1 + m_2} \sqrt{(m_1 v_1)^2 + 2m_1 m_2 v_1 v_2 \cos \alpha + (m_2 v_2)^2}, \]

whereas a division leads to

\[ \tan \beta = \frac{m_2 v_2 \sin \alpha}{m_1 v_1 + m_2 v_2 \cos \alpha}. \]
The loss of mechanical energy during impact is given by the difference $\Delta T$ of the kinetic energies before and after impact:

$$\Delta T = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - \frac{(m_1 + m_2)v^2}{2}$$

$$= \frac{m_1 m_2}{2(m_1 + m_2)} (v_1^2 + v_2^2 - 2v_1 v_2 \cos \alpha).$$

Note that the loss of energy is a maximum if the cars collide head-on ($\alpha = \pi$). The energy $\Delta T$ causes the deformation of the cars.
Example 2.18 A bullet (mass $m$) has a velocity $v_0$ (Fig. 2.27). An explosion causes the bullet to break into two parts (point masses $m_1$ and $m_2$). The directions $\alpha_1$ and $\alpha_2$ of the two parts and the velocity $v_1$ immediately after the explosion are given. Calculate $m_1$ and $v_2$. Determine the trajectory of the center of mass of the two parts.

Solution Since no external forces are acting on the system, its linear momentum does not change (linear momentum before the explosion $=$ linear momentum after the explosion). Component-wise, the principle of conservation of linear momentum gives:

\[ \rightarrow: \quad mv_0 = m_2 v_2 \cos \alpha_2, \]
\[ \uparrow: \quad 0 = m_1 v_1 - m_2 v_2 \sin \alpha_2. \]

Thus, we have two equations for the two unknowns $m_1$ and $v_2$. Solving with $m = m_1 + m_2$ yields

\[ m_1 = m \frac{v_0}{v_1} \tan \alpha_2, \quad v_2 = \cos \alpha_2 - \frac{v_0}{v_1} \sin \alpha_2. \]

We measure the time $t$ from the moment of the explosion. Then, with $|y_i| = v_i t \sin \alpha_i$, the location of the center of mass is given by

\[ y_c = \frac{1}{m} \left( m_1 y_1 + m_2 y_2 \right) \]
\[ = \frac{1}{m} \left( m \frac{v_0}{v_1} \tan \alpha_2 v_1 t + \left( m - m \frac{v_0}{v_1} \tan \alpha_2 \right) \cos \alpha_2 - \frac{v_0}{v_1} \sin \alpha_2 \right) t \]
\[ = 0. \]

The center of mass of the system continues to move on the original straight line $y = 0$ (law of motion for the center of mass).
Example 2.19 A ball (point mass \( m_1 \)) is attached to a cable. It is released from rest at a height \( h_1 \) (Fig. 2.28). After falling to the vertical position at \( A \) it collides with a second ball (point mass \( m_2 = 2m_1 \)) which is also initially at rest. The coefficient of restitution is \( e = 0.8 \).

Determine the height \( h_2 \) which the first ball can reach after the collision and the velocity of the second ball immediately after impact.

Solution The velocities of the masses \( m_1 \) and \( m_2 \) before the impact are

\[
v_1 = \sqrt{2gh_1}, \quad v_2 = 0.
\]

The Impulse Laws

① \( \rightarrow \): \( m_1(v_1 - v_1) = -\hat{F} \),

② \( \rightarrow \): \( m_2v_2 = +\hat{F} \),

and the hypothesis

\[
e = \frac{v_1 - v_2}{v_1}
\]

yield the velocities immediately after the impact:

\[
\overline{v}_1 = v_1 \frac{m_1 - e m_2}{m_1 + m_2} = v_1 \frac{1 - 1.6}{1 + 2} = -0.2 \sqrt{2gh_1},
\]

\[
\overline{v}_2 = v_1(1 + e) \frac{m_1}{m_1 + m_2} = v_1 1.8 \frac{1}{1 + 2} = 0.6 \sqrt{2gh_1}.
\]

The height \( h_2 \) follows from the conservation of energy

\[
\frac{1}{2}m_1\overline{v}_1^2 = m_1gh_2
\]

as

\[
h_2 = \frac{\overline{v}_1^2}{2g} = 0.04h_1.
\]
Example 2.20 The rigid rod (negligible mass) in Fig. 2.29 carries two point masses. It is struck by an impulsive force \( \vec{F} \) at a distance \( a \) from the support \( A \).

Determine the angular velocity of the rod immediately after the impact and the impulsive reaction at \( A \). Calculate \( a \) so that the reaction force at \( A \) is zero.

Solution We introduce a coordinate system. The \( y \)-component of the velocity of the center of mass is zero after the impact. Since the impulsive force \( \vec{F} \) also has no \( y \)-component, the impulsive reaction at \( A \) has only an \( x \)-component, i.e. \( \vec{A}_y = 0 \). We apply the principle of linear impulse and momentum

\[
\rightarrow: 2m(\vec{v}_c - \vec{v}_c) = \vec{A}_x - \vec{F}
\]

and the principle of angular impulse and momentum (compare Chapter 3)

\[
\left( \begin{array}{c}
\vec{A}_y \\
\vec{A}_x \\
\vec{F}
\end{array} \right) = 2 \left( \frac{1}{2} \right)^2 m(\vec{w} - \vec{w}) = \vec{A}_x - \vec{F} \left( \frac{3l}{2} - a \right)
\]

Inserting the kinematic relation

\[
\vec{v}_c = -3l\omega/2
\]

and the velocities

\[
\vec{v}_c = 0 \quad \omega = 0
\]
before the impact we obtain

\[ \bar{\omega} = \frac{\hat{F}a}{5l^2m}, \quad \hat{A}_x = \left(1 - \frac{3a}{5l}\right) \hat{F}. \]

The reaction force at A is zero (\( \hat{A}_x = 0 \)) if the distance \( a \) is chosen as

\[ a = \frac{5l}{3}. \]
Example 2.21  A wagon (weight \(W_1 = m_1g\)) hits a buffer (spring constant \(k\)) with velocity \(v\). The collision causes a block (weight \(W_2 = m_2g\)) to slide on the rough surface (coefficient of static friction \(\mu_0\)) of the wagon (Fig. 2.30).

Determine a lower bound for the wagon’s speed before the collision.

Solution  We separate the wagon and the block. The orientation of the friction force \(H\) is assumed arbitrarily in the free-body diagram. Both bodies have the same acceleration \((\ddot{x}_1 = \ddot{x}_2 = \ddot{x})\) as long as the block does not slide. Therefore, the equations of motion are

\[
\begin{align*}
\left(1\right) & \quad m_1 \ddot{x} = -H - F, \\
\left(2\right) & \quad m_2 \ddot{x} = H.
\end{align*}
\]

With \(F = k x\) we obtain

\[
H = -\frac{m_2}{m_1 + m_2} k x.
\]

The maximum friction force \(H_{\text{max}}\) is obtained at maximal compression \(x_{\text{max}}\) of the spring which follows from the Conservation of Energy Law:

\[
\frac{1}{2} (m_1 + m_2) \dot{x}^2 = \frac{1}{2} k x_{\text{max}}^2 \quad \rightarrow \quad x_{\text{max}} = \sqrt{\frac{m_1 + m_2}{k} v}.
\]
The block is on the verge of slipping if the condition of "limiting friction"

\[ |H_{\text{max}}| = \mu_0 N = \mu_0 m_2 g \quad \rightarrow \quad k \frac{x_{\text{max}}}{m_1 + m_2} = \mu_0 g \]

is satisfied. This yields the velocity that is necessary to cause slippage of the block:

\[ v = \mu_0 g \sqrt{\frac{m_1 + m_2}{k}}. \]
Example 2.22  The system shown in Fig. 2.31 consists of two blocks (masses $m_1$ and $m_2$), a spring (spring constant $k$), a massless rope and two massless pulleys. Block 1 lies on a rough surface (coefficient of kinetic friction $\mu$).

At the beginning of the motion ($t = 0$), the spring is unstretched and the position of block 1 is given by $x_1 = 0$.

Determine the velocity $\dot{x}_1$ as a function of the position $x_1$.

Fig. 2.31

Solution  Since we want to find the velocity as a function of the position, we use the work-energy theorem

$$T_1 - T_0 = U.$$

In moving from the initial position to an arbitrary position, the gravitational force, the force in the spring and the friction force perform work:

$$U_W = m_2gx_2, \quad U_k = -\frac{1}{2}kx_1^2,$$

$$U_R = -Rx_1 = -\mu N x_1 = -\mu m_1gx_1.$$

The kinetic energies are given by

$$T_1 = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2, \quad T_0 = 0.$$

With the kinematic relation

$$\dot{x}_2 = 2\dot{x}_1,$$

the work-energy theorem yields

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = m_2gx_2 - \frac{1}{2}kx_1^2 - \mu m_1gx_1.$$

$$\Rightarrow \quad \dot{x}_1 = \sqrt{\frac{2}{5}(2 - \mu)gx_1 - \frac{1}{5}m_2 \dot{x}_1^2}.$$
Example 3.24 Point $A$ of the rod in Fig. 3.47 moves with constant velocity $v_A$ to the left.

Determine the velocity and the acceleration of point $B$ of the rod (point of contact with the step) as a function of the angle $\phi$. Find the path $y(x)$ of the instantaneous center of rotation.

Solution We use the coordinate system with the basis vectors $e_r$ and $e_\phi$ as shown in the figure. Then, the velocity of point $A$ is given by

$$v_A = v_A \cos \phi e_r - v_A \sin \phi e_\phi.$$ 

The velocity $v_B$ of point $B$ points in the direction of the rod (the rod does not lift off the step). Thus,

$$v_B = v_B e_r.$$ 

With the kinematic relation

$$v_B = v_A + v_{AB}$$ where $v_{AB} = a \dot{\phi} e_\phi$,

and with

$$a = \frac{h}{\sin \phi},$$

we obtain

$$v_B e_r = v_A \cos \phi e_r - v_A \sin \phi e_\phi + \frac{h \dot{\phi}}{\sin \phi} e_\phi \quad \rightarrow \quad v_B = v_A \cos \phi$$ and \(\dot{\phi} = \frac{v_A}{h} \sin^2 \phi\).
In order to determine the acceleration of point $B$ we use the relation
\[ a_B = a_A + a_{AB}' + a_{AB}'' , \]
where
\[ a_A = 0 , \quad a_{AB}' = -a\dot{\varphi}^2 e_r , \quad a_{AB}'' = a\ddot{\varphi}e_{\varphi} . \]

The angular acceleration of the rod is found by differentiating its angular velocity $\dot{\varphi}$:
\[ \ddot{\varphi} = \frac{v_A}{h} (\sin^2 \varphi)^{-1} \quad \rightarrow \quad \ddot{\varphi} = 2\frac{v_A^2}{h^2} \sin^3 \varphi \cos \varphi . \]

Hence, we obtain
\[ a_B = \frac{v_A^2}{h^2} \sin^2 \varphi (-\sin \varphi e_r + 2 \cos \varphi e_{\varphi}) . \]

The instantaneous center of rotation $\Pi$ is given by the point of intersection of two straight lines which are perpendicular to two velocities. In the present example, the directions of the velocities $v_A$ and $v_B$ are known. Therefore, the instantaneous center of rotation can easily be constructed (see the figure).

Point $\Pi$ has the coordinates
\[ x = h \cot \varphi , \quad y = h + x \cot \varphi . \]

Elimination of the angle $\varphi$ yields the path of $\Pi$:
\[ y = \frac{x^2}{h} + h . \]
**Example 3.25** The wheel of a crank drive rotates with constant angular velocity $\omega$ (Fig. 3.48).

Determine the velocity and the acceleration of the piston $P$.

**Solution** If we introduce the auxiliary angle $\psi$, then the position $x_P$ of the piston is given by

$$x_P = r \cos \varphi + l \cos \psi.$$ 

Now we differentiate and obtain (note $\dot{\varphi} = \omega = \text{const}$)

$$\dot{x}_P = -r \omega \sin \varphi - l \dot{\psi} \sin \psi,$$

$$\ddot{x}_P = -r \omega^2 \cos \varphi - l \ddot{\psi} \sin \psi - l \dot{\psi}^2 \cos \psi.$$ 

The quantities $\sin \psi$, $\cos \psi$, $\dot{\psi}$ and $\ddot{\psi}$ are as yet unknown. They follow from the condition that the piston can move only in the horizontal direction. Therefore, its vertical displacement is zero:

$$y_P = 0 = r \sin \varphi - l \sin \psi.$$ 

Differentiation yields

$$\dot{y}_P = 0 = r \omega \cos \varphi - l \dot{\psi} \cos \psi,$$

$$\ddot{y}_P = 0 = -r \omega^2 \sin \varphi - l \ddot{\psi} \cos \psi + l \dot{\psi}^2 \sin \psi.$$
Thus,

\[
\sin \psi = \frac{r}{l} \sin \varphi, \quad \cos \psi = \sqrt{1 - \left(\frac{r}{l} \sin \varphi\right)^2},
\]

\[
\dot{\psi} = \omega \frac{r}{l} \cos \varphi, \quad \ddot{\psi} = -\omega^2 \frac{r}{l} \sin \varphi + \psi^2 \sin \psi \cos \psi.
\]

Hence, we finally obtain

\[
\dot{x}_P = -r\omega \left\{ \sin \varphi + \frac{r}{l} \frac{\sin \varphi \cos \varphi}{\cos \psi} \right\},
\]

\[
\ddot{x}_P = -r\omega^2 \left\{ \cos \varphi - \frac{r}{l} \left[ \frac{\sin^2 \varphi \cos \varphi}{\cos \psi} + \frac{\cos^2 \varphi \cos \psi}{\cos^3 \psi} \right] \right\}.
\]
Example 3.26  Link MA of the mechanism in Fig. 3.49 rotates with angular velocity $\dot{\varphi}(t)$.

Determine the velocities of points B and C, the angular velocity $\omega$ and the angular acceleration $\ddot{\omega}$ of the angled member ABC at the instant shown.

Solution  With the given $x, y, z$-coordinate system, the velocities of points A and B at the instant shown can be written in the form

$$v_A = r \dot{\varphi} \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix}, \quad v_B = v_B \begin{bmatrix} -\cos \alpha \\ \sin \alpha \\ 0 \end{bmatrix},$$

where $v_B$ is as yet unknown (note that the direction of $v_B$ is known). The angular velocity of the angled member ABC is represented by

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}.$$  

We now use the kinematic relation

$$v_B = v_A + \omega \times r_{AB}$$
which, with $\mathbf{r}_{AB} = (0, -l, 0)^T$, in coordinates reads

$$
v_B = r\dot{\varphi} \begin{bmatrix} -\cos \alpha \\ -\sin \alpha \\ 0 \end{bmatrix} + \omega l \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

Solving yields

$$v_B = r\dot{\varphi} \frac{\cos \varphi}{\sin \alpha}, \quad \omega = \frac{r\dot{\varphi}}{l} \left( \sin \varphi - \frac{\cos \varphi}{\tan \alpha} \right).$$

In an analogous way we obtain the velocity of point C:

$$
v_C = v_A + \omega \times \mathbf{r}_{AC} \quad \rightarrow \quad v_C = r\dot{\varphi} \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix}.
$$

In order to determine the angular acceleration $\ddot{\omega} = (0, 0, \ddot{\omega})^T$ of the angled member, we first write down the relation

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\omega} \times \mathbf{r}_{AB} + \omega \times (\omega \times \mathbf{r}_{AB}),$$

which in coordinates reads

$$
a_B = r\ddot{\varphi} \begin{bmatrix} -\cos \alpha \\ -\sin \alpha \\ 0 \end{bmatrix} + r\dot{\varphi}^2 \begin{bmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{bmatrix} + \ddot{\omega} l \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

Solving for $\mathbf{a}_B$ and $\ddot{\omega}$ yields

$$a_B = \frac{1}{\sin \alpha} \left( r\ddot{\varphi} \cos \varphi - r\dot{\varphi}^2 \sin \varphi + \omega^2 l \right),$$

$$\ddot{\omega} = \frac{1}{l} \left[ r\ddot{\varphi} \sin \varphi + r\dot{\varphi}^2 \cos \varphi - 1 \tan \alpha \left( r\ddot{\varphi} \cos \varphi - r\dot{\varphi}^2 \sin \varphi + \omega^2 l \right) \right].$$
Example 3.27  Wheel $\odot$ rolls in a gear mechanism without slip along circle $\odot$. The mechanism is driven with a constant angular velocity $\Omega$ (Fig. 3.50).

Determine the magnitudes of the velocity and the acceleration of point $P$ on the wheel.

Solution  The motion of $P$ is composed of the rotation of $B$ about $A$ with

$$v_B = R\Omega \quad (v_B \perp AB), \quad a_B = R\Omega^2 \quad (a_B \parallel AB)$$

and the rotation of $P$ about $B$. In order to determine the angular velocity $\omega$ of wheel $\odot$ we consider two positions of the wheel. The figure shows that the relation

$$(R + r)\alpha = r(\beta + \alpha)$$

$$\rightarrow \quad R\alpha = r\beta$$

for the arclengths holds. Differentiating and using $\dot{\alpha} = \Omega$ and $\dot{\beta} = \omega$ yields

$$R\Omega = r\omega \quad \rightarrow \quad \omega = \frac{\Omega R}{r}.$$  

Note that $\Omega$ is positive counterclockwise, whereas $\omega$ is positive clockwise.
We can now construct the velocity diagram and the acceleration diagram. Since the velocity diagram is represented by an isosceles triangle, we obtain

\[ v_P = 2\Omega R \sin \frac{\varphi}{2} \, . \]

Finally, the law of cosines yields

\[ a_P^2 = (\Omega^2 R)^2 + \left( \frac{\Omega^2 R^2}{r} \right)^2 - 2\Omega^2 R \frac{\Omega^2 R^2}{r} \cos(\pi - \varphi) \, , \]

\[ a_P = \Omega^2 R \sqrt{1 + \left( \frac{R}{r} \right)^2 + 2 \frac{R}{r} \cos \varphi} \, . \]
Example 3.28  A disk ① (mass $m$, radius $r_1$) rests in a frictionless support ($\omega_0 = 0$). A second disk ② (mass $m$, radius $r_2$) rotates with the angular velocity $\omega_2$. It is placed on disk ① as shown in Fig. 3.51. Due to friction, both disks eventually rotate with the same angular velocity $\bar{\omega}$.

Determine $\bar{\omega}$. Calculate the change $\Delta T$ of the kinetic energy.

Solution  Since there is no moment of external forces acting on the system, the angular momentum is conserved during the process:

$$\Theta_2 \omega_2 = (\Theta_1 + \Theta_2) \bar{\omega}.$$ 

With the mass moments of inertia

$$\Theta_1 = mr_1^2/2, \quad \Theta_2 = mr_2^2/2,$$

we obtain the resulting angular velocity $\bar{\omega}$:

$$\bar{\omega} = \frac{r_2^2}{r_1^2 + r_2^2} \omega_2.$$ 

The change $\Delta T$ of the kinetic energy follows as

$$\Delta T = \frac{1}{2}(\Theta_1 + \Theta_2) \bar{\omega}^2 - \frac{1}{2} \Theta_2 \omega_2^2 \rightarrow \Delta T = -\frac{1}{4} m \omega_2^2 \frac{r_1^2 r_2^2}{r_1^2 + r_2^2}.$$
Example 3.29 The door (mass \(m\), moment of inertia \(\Theta_A\)) of a car is open (Fig. 3.52). Its center of mass \(C\) has a distance \(b\) from the frictionless hinges. The car starts to move with the constant acceleration \(a_0\).

Determine the angular velocity of the door when it slams shut.

Solution First, we write down the acceleration of the center of mass \(C\):

\[
a_C = a_A + a_{AC}^r + a_{AC}^\varphi,
\]

where the magnitudes of the individual terms are given by

\[
a_a = a_0, \quad a_{AC}^r = b\dot{\varphi}^2, \quad a_{AC}^\varphi = b\ddot{\varphi}.
\]

We then formulate the principle of linear momentum (in the \(t\)-direction, see the free-body diagram)

\[
\therefore: \quad m(b\ddot{\varphi} - a_0 \cos \varphi) = A_t
\]

and the principle of angular momentum (about the center of mass \(C\)):

\[
\therefore: \quad \Theta_C \ddot{\varphi} = -A_t b.
\]

The mass moment of inertia \(\Theta_C\) follows from the parallel-axis theorem:

\[
\Theta_C = \Theta_A - mb^2.
\]
Eliminating the force $A_t$ yields

$$\ddot{\varphi} = \frac{ma_0 b}{\Theta_A} \cos \varphi.$$ 

The angular velocity of the door can be obtained through integration:

$$\frac{1}{2} \dot{\varphi}^2(\varphi) = \frac{ma_0 b}{\Theta_A} \sin \varphi.$$ 

Thus, for $\varphi = \pi/2$ (closed door) we find

$$\dot{\varphi}(\pi/2) = \sqrt{\frac{2ma_0 b}{\Theta_A}}.$$
Example 3.30 A child (mass $m$) runs along the rim of a circular platform (mass $M$, radius $r$) starting from point $A$ (Fig. 3.53). The platform is initially at rest; its support is frictionless.

Determine the angle of rotation of the platform when the child arrives again at point $A$.

**Solution** We apply the principle of angular momentum: the time rate of change of the angular momentum is equal to the moment of the applied forces. Since there are no external applied forces acting in the present example, the angular momentum is constant:

\[
\frac{dL^{(0)}}{dt} = 0 \quad \Rightarrow \quad L^{(0)} = \text{const}
\]

with

\[
L^{(0)} = \Theta_0 \omega + mr^2(\omega + \omega_{\text{rel}})
\]

Here,

\[
\Theta_0 = Mr^2/2
\]

is the mass moment of inertia of the platform, $\omega$ is the angular velocity of the platform and $\omega_{\text{rel}}$ is the angular velocity of the child relative to the platform. Integration of the principle of angular momentum with the initial condition

\[
L^{(0)}(0) = 0 \quad \Rightarrow \quad L^{(0)} = 0
\]

yields

\[
\int_0^t L^{(0)} \, dt = 0 \quad \Rightarrow \quad \Theta_0 \varphi + mr^2(\varphi + \varphi_{\text{rel}}) = 0.
\]
The child arrives again at point $A$ when the relative angle attains the value $\varphi_{rel} = 2\pi$. Solving for $\varphi$ yields

$$\varphi = -\frac{2\pi}{\frac{M}{2m} + 1}.$$ 

The result is displayed in a diagram.
Example 3.31  A homogeneous triangular plate of weight \( W = mg \) is suspended from three strings with negligible mass. Determine the acceleration of the plate and the forces in the strings just after string 3 is cut.

Solution  The motion of the plate is a translation after string 3 is cut. Therefore, all the points of the plate have the same acceleration (and the same velocity). We choose point A which rotates about a fixed point to represent the motion of the plate. Its acceleration can be written in the Serret-Frenet frame as

\[
a = a_A = \dot{v} e_t + \frac{v^2}{\rho_A} e_n .
\]

Its velocity is zero at the moment of the cut:

\[
v = 0 \quad \Rightarrow \quad a = \dot{v} e_t .
\]

The principles of linear and angular momentum are then given by (see the free-body diagram)

\[
\bigvee : \quad m \dot{v} = mg \sin \alpha ,
\]

\[
\bigwedge : \quad 0 = S_1 + S_2 - mg \cos \alpha ,
\]

\[
\bigtriangledown : \quad 0 = \frac{2}{3} S_1 \cos \alpha + \frac{1}{3} S_1 \sin \alpha - \frac{1}{3} S_2 \sin \alpha - \frac{4}{3} S_2 \cos \alpha .
\]
Solving these equations yields

\[ \ddot{v} = g \sin \alpha, \]

\[ S_1 = \frac{mg}{6} (\sin \alpha + 4 \cos \alpha), \quad S_2 = \frac{mg}{6} (-\sin \alpha + 2 \cos \alpha). \]

Note that \( S_2 = 0 \) for \( \sin \alpha = 2 \cos \alpha \) (i.e., for \( \tan \alpha = 2 \), \( \alpha = 63.4^\circ \)). In this case the line of action of \( S_1 \) passes through the center of mass \( C \).
Example 3.32  A sphere (mass $m_1$, radius $r$) and a cylindrical wheel (mass $m_2$, radius $r$) are connected by two bars (mass of each bar $m_3/2$, length $l$). They roll down a rough inclined plane (with angle $\alpha$) without slipping (Fig. 3.55).

Find the acceleration of the bars.

Solution  The only external forces that act on the system are the weights of the individual parts. Therefore, conservation of energy

$$ T + V = \text{const} $$

leads to the solution. The kinetic energy is given by

$$ T = (m_1 + m_2 + m_3)\dot{x}^2/2 + (\Theta_1 + \Theta_2)\dot{\phi}^2/2 $$

and the potential energy is

$$ V = -(m_1 + m_2 + m_3)gx \sin \alpha $$

With the kinematic relation (rolling without slipping)

$$ \dot{x} = r\dot{\phi} $$

and the mass moments of inertia

$$ \Theta_1 = 2m_1r^2/5, \quad \Theta_2 = m_2r^2/2 $$
we obtain
\[(7m_1/10 + 3m_2/4 + m_3/2) \ddot{x}^2 - (m_1 + m_2 + m_3)gx \sin \alpha = \text{const} \]

Differentiation finally yields the acceleration of the bar:
\[2(7m_1/10 + 3m_2/4 + m_3/2) \ddot{x} - (m_1 + m_2 + m_3) \dot{x} \sin \alpha = 0 \]
\[\rightarrow a = \ddot{x} = \frac{(m_1 + m_2 + m_3) \dot{x} \sin \alpha}{7m_1/5 + 3m_2/2 + m_3}g\]
Example 3.33  The cylindrical shaft shown in Fig. 3.56 has an inhomogeneous mass density given by $\rho = \rho_0(1 + \alpha r)$.

Find the moments of inertia $\Theta_x$ and $\Theta_y$.

**Solution**  We determine the mass moment of inertia $\Theta_x$ from

$$\Theta_x = \int (y^2 + z^2) dm = \int r^2 dm.$$ 

With

$$dm = \rho 2\pi r \, dr \, dx = 2\pi \rho_0 (r + \alpha r^2) \, dr \, dx,$$

we obtain

$$\Theta_x = 2\pi \rho_0 \int_0^l \int_0^R (r^3 + \alpha r^4) \, dr \, dx = 2\pi \rho_0 l \int_0^R \left( \frac{r^4}{4} + \alpha \frac{r^5}{5} \right) \, dr = \frac{\pi \rho_0 l R^4}{2} \left( 1 + 4 \frac{\alpha R}{5} \right).$$

The mass moment of inertia $\Theta_y = \Theta_z$ (symmetry) follows from

$$\Theta_y = \int (z^2 + x^2) dm$$

with

$$dm = \rho r \, d\varphi \, dr \, dx = \rho_0 (r + \alpha r^2) \, d\varphi \, dr \, dx,$$

$$z = r \sin \varphi.$$
Using the symmetry we obtain

\[ \Theta_y = 4\rho_0 \int_{0}^{R} \int_{0}^{\pi/2} (r^2 \sin^2 \varphi + x^2)(r + \alpha r^2) d\varphi \, dr \, dx \]

\[ = \pi \rho_0 \int_{0}^{R} (r^2 + 2x^2)(r + \alpha r^2) dr \, dx \]

\[ = \pi \rho_0 R^2 l \left[ \frac{R^2}{4} + \frac{R^2}{3} + \alpha \left( \frac{R^3}{5} + \frac{2}{9} R l^2 \right) \right] \]

Note: For \( \alpha = 0 \) the results reduce with \( m = \rho_0 \pi R^2 l \) to the mass moments of inertia of a homogeneous cylindrical shaft (note the term due to the parallel-axis theorem):

\[ \Theta_x = \frac{1}{2} m R^2, \quad \Theta_y = \frac{1}{12} m \left( 3R^2 + 4l^2 \right). \]
Example 3.34 Determine the moment of inertia $\Theta_a$ of a homogeneous torus with a circular cross section and mass $m$.

Solution The mass moment of inertia $\Theta_a$ is determined from

$$\Theta_a = \int r^2 dm.$$ 

We can determine the geometrical relations

$$r = R + c \sin \varphi,$$

$$dm = \rho \frac{2\pi r}{2} c \cos \varphi dr$$

from the figure. With

$$dr = c \cos \varphi d\varphi$$

we obtain

$$dm = 4\pi \rho c^2 (R + c \sin \varphi) \cos^2 \varphi d\varphi.$$ 

Thus,

$$\Theta_a = 4\pi \rho c^2 \int_{-\pi/2}^{+\pi/2} (R + c \sin \varphi)^3 \cos^2 \varphi d\varphi$$

$$= 4\pi \rho c^2 \left[ R^3 \cos^2 \varphi + 3R^2 c \sin \varphi \cos^2 \varphi 
+ 3Rc^2 \sin^2 \varphi \cos^2 \varphi + c^3 \sin^3 \varphi \cos^2 \varphi \right] d\varphi$$

$$= 4\pi \rho c^2 \left( \frac{\pi}{2} R^3 + \frac{3\pi}{8} Rc^2 \right) = 2\pi^2 \rho c^2 R \left( R^2 + \frac{3}{4} c^2 \right).$$
Note that the integrals of the odd functions over the even interval are zero. Using \( m = 2\pi^2\rho c^2 R \) we finally get

\[
\Theta_a = m \left( R^2 + \frac{3}{4} c^2 \right).
\]

This result reduces to \( \Theta_a = m R^2 \) in the case of a thin ring \( (c \ll R) \).
**Example 3.35** A rope drum on a rough surface is set into motion by pulling the rope with a constant force $F_0$.

Determine the acceleration of point $C$ assuming that the drum rolls (no slipping). What coefficient of static friction $\mu_0$ is necessary to ensure rolling?

**Solution** The free-body diagram shows the forces that act on the drum. We apply the principles of linear and angular momentum:

\[ \rightarrow: \quad ma_c = F_0 \cos \alpha - H, \]
\[ \uparrow: \quad 0 = N - mg + F_0 \sin \alpha, \]
\[ \otimes: \quad \Theta \dot{\omega} = r_1 H - r_2 F_0. \]

In addition we have the kinematic relation
\[ v_C = r_1 \omega \quad \rightarrow \quad v_C = a_c = r_1 \dot{\omega}, \]

between the velocity and the angular velocity (rolling without slip). We solve these four equations to obtain the acceleration and the forces:

\[ a_c = \frac{F_0}{m} \cos \alpha \left( \frac{r_2}{r_1} \frac{1}{m} \Theta \right), \]
\[ H = F_0 \frac{r_2^2 m}{r_1 m} \frac{1}{r_1}, \]
\[ N = mg - F_0 \sin \alpha. \]
In order to ensure rolling, the condition of static friction \( H < \mu_0 N \) has to be satisfied. This leads to

\[
\mu_0 \geq \frac{\Theta_c \cos \alpha}{r_1^2 m} + \frac{r_2}{r_1} \left( \frac{mg}{F_0 - \sin \alpha} \right) \left( 1 + \frac{\Theta_c}{r_1^2 m} \right)
\]

Note: For \( \cos \alpha > r_2/r_1 \) we have \( a_c > 0 \) (motion to the right), whereas for \( \cos \alpha < r_2/r_1 \) we obtain \( a_c < 0 \) (motion to the left). In the case of \( F_0 \sin \alpha > mg \) the drum lifts off the ground.
Example 3.36  A homogeneous beam (mass $M$, length $l$) is initially in vertical position $\odot$ (Fig. 3.59). A small disturbance causes the beam to rotate about the frictionless support $A$ (initial velocity equal to zero). In position $\Box$ it strikes a small sphere (mass $m$, radius $r \ll l$). Assume the impact to be elastic ($e = 1$).

Determine the angular velocities of the beam immediately before and after the impact and the velocity of the sphere after the impact.

Solution  The angular velocity $\omega$ immediately before the impact follows from the conservation of energy ($\Theta_A = Ml^2/3$):

$$Mgl/2 = \Theta_A \omega^2/2 - Mgl/2 \rightarrow \omega = \sqrt{6g/l}$$

Since the impact is assumed to be elastic, the conservation of the angular momentum

$$\Theta_A \omega = \Theta_A \bar{\omega} + ml \bar{v}$$

as well as the conservation of energy

$$\Theta_A \omega^2/2 = \Theta_A \bar{\omega}^2/2 + ml^2/2$$

hold. Solving these two equations yields

$$\bar{\omega} = \frac{M - 3m}{M + 3m} \omega, \quad \bar{v} = \frac{2M}{M + 3m} \omega$$
Example 3.37 A thin half-cylindrical shell of weight $W = mg$ rolls without sliding on a flat surface (Fig. 3.60).

Determine the angular velocity as a function of $\varphi$ when the initial condition $\dot{\varphi}(\varphi = 0) = 0$ is given.

Solution The position of the center of mass and the mass moment of inertia are given by

$$
c = \frac{1}{R} \int_0^\pi (R \sin \alpha) R d\alpha = \frac{2R}{\pi},
$$

$$
\Theta_c = \Theta_0 - mc^2 = mR^2 - \frac{4}{\pi^2}mR^2
$$

$$
= mR^2(1 - \frac{4}{\pi^2}).
$$

Thus,

$$
x_c = R\varphi + c \cos \varphi = R(\varphi + \frac{2}{\pi} \cos \varphi),
$$

$$
y_c = c \sin \varphi = \frac{2R}{\pi} \sin \varphi,
$$

$$
\dot{x}_c = R\dot{\varphi}(1 - \frac{2}{\pi} \sin \varphi),
$$

$$
\dot{y}_c = \frac{2R}{\pi} \dot{\varphi} \cos \varphi.
$$

Since we want to find the angular velocity as a function of the angle we apply the conservation of energy:

$$
T + V = T_0 + V_0.
$$

With

$$
T_0 = 0, \quad V_0 = 0,
$$

$$
T = \frac{1}{2} m(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2} \Theta_c \dot{\varphi}^2 = \frac{1}{2} mR^2 \dot{\varphi}^2 \left(1 - \frac{4}{\pi^2} \sin \varphi + \frac{4}{\pi^2} \sin^2 \varphi + \frac{4}{\pi^2} \cos^2 \varphi\right) + \frac{1}{2} mR^2 \dot{\varphi}^2 \left(1 - \frac{4}{\pi^2}\right) = mR^2 \dot{\varphi}^2 \left(1 - \frac{2}{\pi} \sin \varphi\right),
$$

$$
V = -mgyc = -\frac{2}{\pi}mgR \sin \varphi.$$
we obtain

\[ mR^2 \dot{\varphi}^2 (1 - \frac{2}{\pi} \sin \varphi) - \frac{2}{\pi} m g r \sin \varphi = 0 \]

\[ \rightarrow \dot{\varphi}(\varphi) = \frac{2g \sin \varphi}{R(\pi - 2 \sin \varphi)} . \]

Note that the angular velocity attains its maximum for \( \varphi = 90^\circ \) (lowest position of the center of mass):

\[ \dot{\varphi}_{\text{max}} = \sqrt{\frac{2g}{R(\pi - 2)}} . \]
Example 3.38  An elevator consists of a cabin (weight $W = mg$) which is connected through a rope (of negligible mass) with a rope drum and a band brake (coefficient of dynamic friction $\mu$).

Determine the necessary braking force $F$ such that a cabin travelling downwards with velocity $v_0$ stops after a distance $h$.

Solution  First we determine the brake momentum $M_B$ that acts on the drum. To this end we separate the drum and the lever. Equilibrium at the lever

$$\begin{align*}
\dot{A} : \quad 0 &= -2r_2 S_2 + l F \\
\Rightarrow S_2 &= \frac{Fl}{2r_2}
\end{align*}$$

and the formula $S_1 = S_2 e^{-\mu \pi}$ for belt friction (see Volume 1, Chapter 9) lead to

$$M_B = r_2 S_2 - r_1 S_1 = (1 - e^{-\mu \pi}) \frac{Fl}{2}.$$ 

Now we apply the work-energy theorem

$$T_1 - T_0 = U.$$
With the kinematic relations (constraints imposed by the rope)
\[ v_0 = r_1 \omega_0, \quad h = r_1 \varphi_h \]
the kinetic energies and the work are given by
\[
T_1 = 0, \quad T_0 = \frac{1}{2} m v_0^2 + \frac{1}{2} \Theta_0 \omega_0^2 = \frac{v_0^2}{2} \left( m + \frac{\Theta_0}{r_1^2} \right),
\]
\[
U = mgh - \int_0^{\varphi_h} M_B d\varphi = mgh - \left( 1 - e^{-\mu \pi} \right) \frac{F h}{2r_1^2}.
\]
Thus, we obtain
\[
\frac{v_0^2}{2} \left( m + \frac{\Theta_0}{r_1^2} \right) = mgh - \left( 1 - e^{-\mu \pi} \right) \frac{F h}{2r_1^2}.
\]
Solving for the force \( F \) yields
\[
F = \frac{r_1 v_0^2 m}{l h (1 - e^{-\mu \pi})} \left( 1 + \frac{\Theta_0}{m r_1^2} + \frac{2gh}{v_0^2} \right).
\]
**Example 3.39** A homogeneous beam (mass \( m \), length \( l \)) rotates about frictionless support \( A \) (Fig. 3.62) until it hits support \( B \). The motion starts with zero initial velocity in the vertical position. The coefficient of restitution \( \epsilon \) is given.

Calculate the impulsive forces at \( A \) and \( B \). Determine the distance \( a \) so that the impulsive force at \( A \) vanishes. Calculate the change of the kinetic energy.

**Solution** First we calculate the angular velocity of the beam immediately before the impact with the aid of the conservation of energy \((\Theta_A = ml^2/3)\):

\[
mgl/2 = \Theta_A \dot{\phi}^2/2 \quad \rightarrow \quad \dot{\phi}^2 = \frac{3g}{l}.
\]

We then apply the principles of impulse and momentum:

\( \rightarrow \): \( m(\ddot{x}_C - x_C) = \hat{A}_H \),

\( \uparrow \): \( m(\ddot{y}_C - y_C) = \hat{A}_V + \hat{B} \),

\( \wedge \): \( \Theta_A(\ddot{\phi} - \dot{\phi}) = -\hat{B}a \).
Here, the velocity of the center of mass and the angular velocity before the impact are given by
\[ \dot{x}_C = 0, \quad \dot{y}_C = -\sqrt{3}g/2, \quad \dot{\phi} = \sqrt{3}g/l. \]

In order to obtain the velocities immediately after the impact we use the hypothesis
\[ e = -\frac{\ddot{y}_B}{\ddot{y}_B} \rightarrow \ddot{\phi} = -e\dot{\phi}, \]
which leads to
\[ \dot{x}_C = 0, \quad \dot{y}_C = e\sqrt{3}g/l/2, \quad \dot{\phi} = -e\sqrt{3}g/l. \]

Solving for the impulsive reactions yields
\[ \hat{A}_H = 0, \quad \hat{A}_V = \frac{(1 + e)(3a - 2l)m}{6a\sqrt{3}gl}, \quad \hat{B} = (1 + e)ml \frac{3a}{3a - \sqrt{3}gl}. \]

The impulsive force at A vanishes if
\[ \hat{A}_V = 0 \rightarrow a = 2l/3. \]

The change of the kinetic energy is the difference \( \Delta T = T_1 - T_0 \) of the energies before and after impact:
\[ \Delta T = \frac{1}{2} \Theta A \dot{\phi}^2 - \frac{1}{2} \Theta A \dot{\phi}^2 \rightarrow \Delta T = -(1 - e^2)mg/l/2. \]
Example 3.40 A homogeneous circular disk of weight $W = mg$ is suspended from a pin-supported bar (of negligible mass). Initially the disk rotates with the angular velocity $\omega_0$.

a) Determine the amplitude of oscillation of the pendulum, if the bar suddenly prevents the disk from rotating.

b) Calculate the energy loss $\Delta E$.

Solution a) First we determine the angular velocity $\omega$ of the bar immediately after the rotation of the disk is prevented (note that the bar is still in the vertical position). Since there are no external moments acting with respect to $A$, the angular momentum is conserved. With

$$\Theta_B = \frac{1}{2} mr^2, \quad \Theta_A = \frac{1}{2} mr^2 + ml^2 = \frac{1}{2} m(r^2 + 2l^2)$$

we obtain

$$\Theta_B \omega_0 = \Theta_A \omega \quad \rightarrow \quad \omega = \frac{\Theta_B}{\Theta_A} \omega_0 = \frac{r^2}{r^2 + 2l^2} \omega_0.$$ 

The maximum value $\varphi_1$ of the angle $\varphi$ (= amplitude of the oscillation) can be calculated from the conservation of energy:

$$T_1 + V_1 = T_0 + V_0.$$ 

If we choose the potential energy $V_0$ (initial position) to be equal to zero, we can write

$$T_1 = 0,$$

$$V_1 = mg(l - l \cos \varphi_1),$$

$$T_0 = \frac{1}{2} \Theta_A \omega^2 = \frac{mr^2 \omega_0^2}{4} \frac{r^2}{r^2 + 2l^2},$$

$$V_0 = 0.$$
Then, conservation of energy leads to
\[
\cos \varphi_1 = 1 - \frac{r^2 \omega_0^2}{4gl} \frac{r^2}{r^2 + 2l^2}.
\]

b) The loss of energy \( \Delta E \) is given by the difference \( \Delta T \) of the kinetic energies before and after the blocking:
\[
\Delta E = \frac{\Theta_B \omega_0^2}{2} - \frac{\Theta_A \omega^2}{2} = \frac{mr^2 \omega_0^2}{4} - \frac{m}{4} \frac{(r^2 + 2l^2) r^4 \omega_0^2}{(r^2 + 2l^2)^2} = \frac{mr^2 \omega_0^2}{2} \frac{l^2}{r^2 + 2l^2}.
\]
**Example 3.41** A homogeneous angled bar of mass \(m\) is attached to a shaft with negligible mass. The rotation of the system is driven by the moment \(M_0\).

Determine the angular acceleration and the support reactions.

**Solution** The following moments and products of inertia with respect to the body-fixed coordinate system \(\xi, \eta, \zeta\) are needed:

\[
\Theta_\zeta = \frac{2}{3} m \left(\frac{2l}{3}\right)^2 + \frac{m}{3}\left(2l\right)^2 = \frac{20}{9} ml^2,
\]
\[
\Theta_{\xi\zeta} = -\frac{m}{3} \frac{2l}{2} l = -\frac{1}{3} ml^2,
\]
\[
\Theta_{\eta\zeta} = 0.
\]

With the moments

\[
M_\xi = 2lB_\eta - 2lA_\eta, \quad M_\eta = 2lA_\xi - 2lB_\xi, \quad M_\zeta = M_0
\]

the principle of angular momentum in component form yields

\[
\vec{\xi} : 2l(B_\eta - A_\eta) = -\omega \frac{ml^2}{3} \rightarrow B_\eta - A_\eta = -\frac{ml\omega}{6},
\]
\[
\vec{\eta} : 2l(A_\xi - B_\xi) = \omega \frac{2ml^2}{3} \rightarrow A_\xi - B_\xi = -\frac{ml\omega^2}{6},
\]
\[
\vec{\zeta} : M_0 = \frac{20}{9} ml^2 \omega \rightarrow \dot{\omega} = \frac{9M_0}{20ml^2}.
\]
In order to be able to calculate the support reactions, we now have to formulate the principle of linear momentum. The center of mass moves on a circle. With the distance $\xi_c = 4l/3$ from the axis of rotation, we obtain the components of its acceleration as $a\xi_c = -\xi_c \omega^2$ and $a\eta_c = \xi_c \dot{\omega}$. Thus,

$$-m \xi_c \omega^2 = A_\xi + B_\xi, \quad m \xi_c \dot{\omega} = A_\eta + B_\eta,$$

which leads to

$$A_\xi = -\frac{3}{4} ml \omega^2, \quad A_\eta = \frac{27}{80} \frac{M_0}{l},$$

$$B_\xi = \frac{7}{12} ml \omega^2, \quad B_\eta = \frac{21}{80} \frac{M_0}{l}.$$
Example 3.42 A shaft (principal moments of inertia $\Theta_1$, $\Theta_2$, $\Theta_3$) rotates with constant angular velocity $\omega_0$ about its longitudinal axis. This axis undergoes a rotation $\alpha(t)$ about the z-axis of the fixed in space system $x, y, z$.

Calculate the moment which is exerted by the bearings on the shaft for
a) uniform rotation $\alpha = \Omega t$,
b) harmonic rotation $\alpha = \alpha_0 \sin \Omega t$.

Solution

We solve the problem with the aid of Euler’s equations

$$
\Theta_1 \dot{\omega}_1 - (\Theta_2 - \Theta_3) \omega_2 \omega_3 = M_1,
$$

$$
\Theta_2 \dot{\omega}_2 - (\Theta_3 - \Theta_1) \omega_3 \omega_1 = M_2,
$$

$$
\Theta_3 \dot{\omega}_3 - (\Theta_1 - \Theta_2) \omega_1 \omega_2 = M_3.
$$

With

$$
\omega_1 = \omega_0, \quad \dot{\omega}_1 = 0, \quad \omega_2 = \dot{\omega}_2 = 0, \quad \omega_3 = \dot{\alpha}, \quad \dot{\omega}_3 = \ddot{\alpha}
$$

we obtain

$$
M_1 = 0, \quad M_2 = - (\Theta_3 - \Theta_1) \omega_0 \dot{\alpha}, \quad M_3 = \Theta_3 \ddot{\alpha}
$$

where $\alpha(t)$ represents an arbitrary rotation.

a) In the special case of a uniform rotation $\alpha = \Omega t$ we get

$$
\dot{\alpha} = \Omega, \quad \ddot{\alpha} = 0
$$

Thus,

$$
M_1 = M_3 = 0, \quad M_2 = (\Theta_1 - \Theta_3) \omega_0 \Omega.
$$
b) In the case of a harmonic rotation $\alpha = \alpha_0 \sin \Omega t$ we find

\[ \dot{\alpha} = \alpha_0 \Omega \cos \Omega t, \quad \ddot{\alpha} = -\alpha_0 \Omega^2 \sin \Omega t \]

and

\[ M_1 = 0, \quad M_2 = (\Theta_1 - \Theta_3) \omega_0 \Omega \alpha_0 \cos \Omega t, \quad M_3 = -\Theta_3 \Omega^2 \alpha_0 \sin \Omega t. \]

Note: Only a moment about the 2-axis acts in case a). It is caused by two opposite support reactions of equal magnitude (= couple) in the 3- and z-directions, respectively.
Example 3.43 A pin-supported rigid beam (mass \( m \), length \( l \)) is initially at rest. At time \( t_0 = 0 \) it starts to rotate due to an applied constant moment \( M_0 \).

Determine the stress resultants (internal forces and moments) as functions of \( x \) for \( t > t_0 \). Neglect gravitational effects.

Solution First we determine the angular acceleration and the angular velocity of the beam. The principle of angular momentum with respect to \( A \) yields

\[
\Theta_A \ddot{\phi} = M_0 \quad \rightarrow \quad \ddot{\phi} = \frac{M_0}{\Theta_A}.
\]

Integration with the initial condition \( \dot{\phi}(0) = 0 \) gives

\[
\dot{\phi}(t) = \frac{M_0}{\Theta_A} t.
\]

Now we cut the beam at an arbitrary position and introduce the bending moment \( M \) and the shear force \( V \) into the free-body diagram (the normal force \( N \) will be considered later). The principle of angular momentum (with respect to the center of mass \( \bar{C} \)) and the principle of linear momentum (in the \( y \)-direction) yield

\[
\bar{C} : \quad \bar{\Theta}_\bar{C} \ddot{\phi} = -M(x) - V(x) \frac{l + x}{2},
\]

\[
\bar{m} \ddot{y}_\bar{C} = V(x).
\]

The acceleration \( \ddot{y}_\bar{C} \) follows from the kinematics (circular motion):

\[
\ddot{y}_\bar{C} = r_{\bar{C}} \ddot{\phi} = \frac{l + x}{2} \ddot{\phi}.
\]

Thus, with \( \bar{m} = \left(1 + \frac{x}{l}\right) m \) and \( \Theta_A = \frac{ml^2}{3} \) we obtain the shear force

\[
V(x) = \bar{m} \frac{l + x}{2} \ddot{\phi} = \bar{m} \frac{l}{2 \Theta_A} \frac{M_0}{l} \left[1 - \left(\frac{x}{l}\right)^2\right].
\]
Introduction of the moment of inertia $\bar{\Theta}_C = \frac{1}{12} \bar{m} (l - x)^2$ leads to the bending moment:

\[
M(x) = -V \frac{l-x}{2} - \bar{\Theta}_C \ddot{\phi}
\]
\[
= -\frac{3}{4} M_0 \left(1 - \frac{x}{l}\right)^2 \left(1 + \frac{x}{l}\right) - \frac{m l^2}{12} \left(1 - \frac{x}{l}\right) \frac{M_0}{\Theta_A}
\]
\[
= -M_0 \left(1 - \frac{x}{l}\right)^2 \left(1 + \frac{1}{2} \frac{x}{l}\right).
\]

The normal force can be determined from the equation of motion in the $x$-direction:

\[
\vec{m} \ddot{x}_C = -N(x)
\]

where $\ddot{x}_C = -r_C \ddot{\phi}^2$ is the centripetal acceleration. This leads to

\[
N(x) = \bar{m} r_C \ddot{\phi}^2
\]
\[
= m \left(1 - \frac{x}{l}\right) \frac{x + l}{2} \left(\frac{M_0}{\Theta_A}\right)^2
\]
\[
= \frac{9}{2} \frac{M_0^2 l^2}{m l^3} \left[1 + \left(\frac{x}{l}\right)^2\right].
\]

Note that the normal force increases with time $t$ in contrast to the bending moment and the shear force.

The stress resultant diagrams are presented below.
Example 3.44  The angled member (weight $W = mg$) in Fig. 3.67 consists of two homogeneous bars. Derive the equation of motion for the member’s center of mass.

Solution  First, we locate the center of mass of the angled member. With the coordinate system as shown in the figure we obtain

\[
x_C = \frac{m l}{3 \cdot \frac{2}{3}} = \frac{l}{6} \\
y_C = \frac{m}{3} \left( 2l + \frac{2m}{3} l \right) = \frac{4}{3} l 
\]

Thus, the distance of the center of mass from point $A$ is given by

\[
a = \sqrt{y_C^2 + x_C^2} = \sqrt{\frac{65}{6}} l
\]

The member rotates about a fixed axis that passes through $A$. We introduce the angle $\phi$ and apply the principle of angular momentum:

\[
\Theta_A \ddot{\phi} = M_A
\]
We measure $\varphi$ from the position of equilibrium ($C$ is vertically below $A$). With the mass moment of inertia

$$\Theta_A = \frac{m}{3} \left\{ \frac{l^2}{12} + \left[ (2l)^2 + \left( \frac{l}{2} \right)^2 \right] \right\} + \frac{2}{3} m \left( \frac{2l}{3} \right)^2 = \frac{7}{3} m l^2$$

we obtain

$$\overset{\circ}{A}: \quad \frac{7}{3} m l^2 \ddot{\varphi} = -m g a \sin \varphi \quad \Rightarrow \quad \ddot{\varphi} + \frac{\sqrt{65}}{14} \frac{g}{l} \sin \varphi = 0.$$

Note: In the case of small oscillations ($\varphi \ll 1$, $\sin \varphi \approx \varphi$) the equation of motion reduces to the differential equation

$$\ddot{\varphi} + \frac{\sqrt{65}}{14} \frac{g}{l} \varphi = 0$$

for harmonic vibrations.
**Example 3.45** A bowling ball (mass $m$) is placed on a rough surface (coefficient of kinetic friction $\mu = 0.3$) with velocity $v_0 = 5$ m/s (Fig. 3.68). Initially, the ball does not rotate.

What is the position $x_r$ of the ball when it stops sliding? Calculate the corresponding velocity $v_r$.

**Solution** When the ball is placed on the rough surface it slides.

The friction force $R$ is opposed to the direction of the motion. Thus, the equations of motion are given by:

\[
\begin{align*}
\rightarrow : & \quad m\ddot{x} = -R, \\
\uparrow : & \quad 0 = N - mg \quad \Rightarrow \quad N = mg, \\
\rightarrow C : & \quad \Theta_c \dot{\omega} = rR.
\end{align*}
\]

With $\Theta_c = 2mr^2/5$ and the law of friction $R = \mu N = \mu mg$, the first and the third equation lead to (initial conditions $v(t=0) = v_0$, $x(t=0) = 0$, $\dot{\omega}(t=0) = 0$)

\[
\begin{align*}
v &= \dot{x} = v_0 - \mu gt, \\
x &= v_0 t - \frac{1}{2} \mu gt^2, \\
\omega &= \frac{5\mu g}{2r} t.
\end{align*}
\]
The ball rolls without sliding if the velocity of its center of mass is given by

\[ v = r \omega . \]

This condition leads to the corresponding time \( t_r \):

\[ v_0 - \mu g t_r = \frac{5}{2} \mu g t_r \quad \rightarrow \quad t_r = \frac{2v_0}{7\mu g} = 0.49 \text{ s}. \]

Thus,

\[ x_r = x(t_r) = \frac{12v_0^2}{49\mu g} = 2.08 \text{ m}, \]

\[ v_r = v(t_r) = \frac{5}{7} v_0 = 3.57 \text{ m/s}. \]
Example 3.46  The double pendulum in Fig. 3.69 consists of two identical homogeneous bars (each mass $m$, length $l$). It is struck by a linear impulse $\hat{F}$ at point $D$.

Determine the distance $d$ of point $D$ from the lower end of the pendulum so that the angular velocity $\omega_2$ of the lower bar is zero immediately after the impact. Calculate the impulsive forces at $A$ and $B$.

Solution  We separate the two bars and draw the free-body diagram. Note that there is no linear impulse in the vertical direction. The bars are at rest before the impact. The principles of linear and angular impulse and momentum are given by

$\Theta_A \omega_1 = l B_1$,

$m \omega_2 = \hat{F} - B$,

$\Theta_C \omega_2 = \frac{l}{2} B - (d - \frac{l}{2}) \hat{F}$

where

$\Theta_A = \frac{1}{3} ml^2$,

$\Theta_C = \frac{1}{12} ml^2$.

We desire the motion of bar $C$ to be a translation. Therefore,

$\omega_2 = 0$,  $\omega_2 = l \omega_1$. 
This leads to

\[ \omega_1 = \frac{3}{4} \frac{\hat{F}}{ml}, \quad \hat{B} = \frac{\hat{F}}{1 + \frac{ml^2}{\Theta_A}}, \quad \hat{F} = \frac{\hat{F}}{4}, \quad d = \frac{l}{2} \frac{2 + \frac{ml^2}{\Theta_A}}{1 + \frac{ml^2}{\Theta_A}} = \frac{5}{8}. \]

The impulsive force at A follows from the principle of linear impulse for bar \( \hat{A} \) with the kinematic relation \( \vec{v}_1 = \frac{1}{2} \vec{e}_1 \):

\[ \rightarrow : \quad m \vec{v}_1 = \hat{A} + \hat{B} \rightarrow \hat{A} = \frac{\hat{F}}{8}. \]
Chapter 4

Principles of Mechanics
**Example 4.8** A homogeneous disk (mass $m$, radius $r$) rolls without slipping on a rough surface (Fig. 4.9). Its center of mass $C$ is connected with the wall by a spring (spring constant $k$).

Derive the equation of motion using

a) Newton’s 2nd Law,
b) dynamic equilibrium conditions.

**Solution**

a) The free-body diagram shows the forces that act on the disk.

We use the coordinates $x$ and $\phi$. Then the principles of linear and angular momentum yield

- $m\ddot{x}_c = -F - H$,
- $0 = N - W \rightarrow N = W$,
- $\Theta_C \ddot{\phi} = H$ where $\Theta_C = mr^2/2$.

In addition, we have the kinematic relation

$\dot{x}_c = r\dot{\phi} \rightarrow \ddot{x}_c = r\ddot{\phi}$,

and the relation

$F = kr\phi$
for the force $F$ in the spring. Solving these equations for the angle $\varphi$ yields

$$\ddot{\varphi} + \omega^2 \varphi = 0 \quad \text{where} \quad \omega^2 = \frac{2k}{3m}.$$  

b) If we apply the dynamic equilibrium conditions, the inertial force $m\dddot{x}_c$ and the pseudo moment $\Theta_C \ddot{\varphi}$ (both acting in the negative coordinate directions) have to be drawn into the free-body diagram.

Dynamic moment equilibrium about point $B$ then yields

$$\hat{B}: \quad \Theta_C \ddot{\varphi} + rm\dddot{x}_c + rF = 0.$$  

If we introduce the kinematic relation (see a)), the force $F = kr\varphi$ and $\Theta_C = mr^2/2$ we again obtain

$$\ddot{\varphi} + \omega^2 \varphi = 0, \quad \omega^2 = \frac{2k}{3m}.$$  

Note that we may choose point $B$ to be the reference point for the moment equilibrium. This is advantageous since then the lever arm of the unknown force of static friction $H$ is zero. The mass moment of inertia must be $\Theta_C$ (not $\Theta_B$!).
Example 4.9 A cylinder (mass \( m \), radius \( r \)) rolls without slipping on a circular path (radius \( R \)); see Fig. 4.10.

Derive the equation of motion using dynamic equilibrium conditions.

Solution We isolate the cylinder and introduce the coordinates \( \varphi \) and \( \psi \) (angle of rotation of the cylinder). With the tangential acceleration \( a_t = (R - r) \ddot{\varphi} \) (in the positive \( \varphi \)-direction) of the center of mass \( C \) and the normal acceleration \( a_n = (R - r) \ddot{\varphi}^2 \) (directed towards the center of the circular path), the inertial forces \( ma_t \) (opposite to \( a_t \)) and \( ma_n \) (opposite to \( a_n \)) can be drawn on the free-body diagram.

The pseudo moment \( \Theta_C \ddot{\psi} \) acts in the negative \( \psi \)-direction. Moment equilibrium about point \( B \) yields the equation of motion:

\[
\hat{B}: \quad \Theta_C \ddot{\psi} + m(R - r) \ddot{\varphi} + mgr \sin \varphi = 0 \quad \text{where} \quad \Theta_C = mr^2/2 .
\]

The system has one degree of freedom. Therefore a relation exists between the two coordinates:

\[
v_C = (R - r) \dot{\varphi} = \dot{\psi} \quad \rightarrow \quad \ddot{\psi} = (R/r - 1) \ddot{\varphi} .
\]

This leads to

\[
\ddot{\varphi} + \frac{2g}{3(R - r)} \sin \varphi = 0 .
\]

Note that the moment equilibrium may be established with respect to point \( B \). This is advantageous since the lever arms of the unknown force of static friction \( H \) and of the inertial force \( m(R - r) \dot{\varphi}^2 \) are zero. The mass moment of inertia, however, has to be taken with respect to the center of mass \( C \).
Example 4.10 Two blocks of weights $W_1 = m_1g$ and $W_2 = m_2g$ are suspended by a pin-supported rope drum (moment of inertia $\Theta_A$) as shown in Fig. 4.11.

Determine the angular acceleration of the drum and the force in rope $\Omega$ using dynamic equilibrium conditions. Neglect the mass of the ropes.

Solution We first introduce the coordinates $x_1$ and $x_2$ describing the motion of the blocks. The inertial forces $-m_i\ddot{x}_i$ point in the negative $x_i$-directions (see the free-body diagram). In addition, we have to consider the pseudo moment $-\Theta_A\ddot{\varphi}$ which acts in the negative $\varphi$-direction. Moment equilibrium about point $A$ then yields

$$A : -r_1m_1(g + \ddot{x}_1) + r_2m_2(g - \ddot{x}_2) - \Theta_A\ddot{\varphi} = 0.$$ 

Using the kinematic relations

$$x_1 = r_1\varphi \quad \Rightarrow \quad \ddot{x}_1 = r_1\ddot{\varphi},$$
$$x_2 = r_2\varphi \quad \Rightarrow \quad \ddot{x}_2 = r_2\ddot{\varphi},$$

we obtain the angular acceleration of the drum:

$$\ddot{\varphi} = \frac{r_2m_2 - r_1m_1}{r_1m_1 + r_2m_2 + \Theta_A} g.$$
In order to determine the force in rope ① we cut the rope. Force equilibrium (see the free-body diagram) yields

\[ S_1 - m_1g - m_1\ddot{x}_1 = 0 \]

or

\[ S_1 = m_1(g + r_1\ddot{\varphi}) = m_1g \frac{r_2(r_1 + r_2)m_2 + \Theta_A}{r_1^2m_1 + r_2^2m_2 + \Theta_A} \]

Note: For \( r_2m_2 > r_1m_1 \) the drum rotates clockwise, for \( r_2m_2 < r_1m_1 \) it rotates counterclockwise. In the special case \( r_2m_2 = r_1m_1 \) the system is in static equilibrium (\( \ddot{\varphi} = 0 \)).
Example 4.11 An angled arm (mass \( m \)) rotates with constant angular velocity \( \Omega \) about point 0 (Fig. 4.12).

Calculate the bending moment, shear force and normal force as functions of position using dynamic equilibrium conditions.

Solution We introduce two coordinate systems \( x_i, z_i \). Then, we make a cut at the arbitrary position \( x_1 \). The acceleration of the mass element \( dm \) at the position \( s \) (distance from the left end of the arm) is given by \( a_n = r\Omega^2 \) (pointing towards point 0; note that \( a_t = 0 \)). Therefore, this element is subjected to the inertial force \( dmr\Omega^2 \) (see the free-body diagram).

With
\[
 dm = \frac{m}{a + b} \, ds = \mu \, ds,
\]
where \( \mu = m/(a + b) \) is the mass per unit length, and with the geometrical relations
\[
 \cos \varphi = (b - s)/r, \quad \sin \varphi = a/r
\]
we can determine the stress resultants through integration (note that \( M' = V \)).

Normal force \((0 \leq x_1 \leq b)\):
\[
 N(x_1) = \int r\Omega^2 \cos \varphi \, dm = \int_0^{x_1} \Omega^2 (b - s) \mu \, ds = \mu \Omega^2 [bs - s^2/2]_0^{x_1} \rightarrow N(x_1) = \mu \Omega^2 (bx_1 - x_1^2/2).
\]

Shear force \((0 \leq x_1 \leq b)\):
\[
 V(x_1) = \int r\Omega^2 \sin \varphi \, dm = \int_0^{x_1} \Omega^2 a \mu \, ds \rightarrow V(x_1) = \mu \Omega^2 ax_1.
\]
Bending moment \((0 \leq x_1 \leq b)\):

\[
M(x_1) = \int_0^{x_1} V(s)\,ds \quad \text{→} \quad M(x_1) = \mu \Omega^2 a x_1^2 / 2.
\]

The matching conditions at the corner \((x_1 = b, x_2 = 0)\) are given by

\[
N_0 = N(x_2 = 0) = V(x_1 = b) = \mu \Omega^2 ab,
\]

\[
V_0 = V(x_2 = 0) = -N(x_1 = b) = -\mu \Omega^2 b^2 / 2,
\]

\[
M_0 = M(x_2 = 0) = M(x_1 = b) = \mu \Omega^2 ab^2 / 2.
\]

Now we make a cut at the position \(x_2\). The mass element \(dm\) at the position \(s\) is subjected to the inertial force \(dm(a - s)\Omega^2\). This leads to the following stress resultants:

**Normal force, shear force, bending moment \((0 \leq x_2 \leq a)\):**

\[
N(x_2) = N_0 + \int_0^{x_2} \mu \Omega^2 (a - s)\,ds
\]

\[
\text{→} \quad N(x_2) = \mu \Omega^2 (ab + ax_2 - x_2^2 / 2).
\]

\[
V(x_2) = V_0 \quad \text{→} \quad V(x_2) = -\mu \Omega^2 b^2 / 2.
\]

\[
M(x_2) = M_0 + x_2 V_0 \quad \text{→} \quad M(x_2) = \mu \Omega^2 b^2 (a - x_2) / 2.
\]
Example 4.12 A wheel (weight $W_1 = m_1 g$, moment of inertia $\Theta_A$) on an inclined plane is connected to a block (weight $W_2 = m_2 g$) by a rope which is guided over an ideal pulley (Fig. 4.13). The wheel rolls on the plane without slipping.

Determine the acceleration of the block applying d’Alembert’s principle. Neglect the masses of the rope and the pulley.

Solution Since the constraint forces (force in the rope, static friction force) need not be determined, it is advantageous to apply d’Alembert’s principle. The motion is described by the coordinates $x_i$ and $\phi$. The inertial forces $m_i \ddot{x}_i$ and the pseudo moment $\Theta_A \ddot{\phi}$ (acting in the directions opposite to the chosen positive coordinate directions) are shown in the figure. D’Alembert’s principle (principle of virtual work) requires that the virtual work of all forces vanishes:

$$\delta U + \delta U_1 = 0$$

$$\rightarrow -m_1 \dddot{x}_1 \delta x_1 - m_1 g \sin \alpha \delta x_1 - \Theta_A \ddot{\phi} \delta \phi + m_2 g \delta x_2 - m_2 \dddot{x}_2 \delta x_2 = 0.$$ 

With the kinematic relations

$$x_1 = x_2 = r \phi = x$$

$$\delta x_1 = \delta x_2 = r \delta \phi = \delta x$$

$$\dddot{x}_1 = \dddot{x}_2 = r \ddot{\phi} = \dddot{x}$$

we obtain

$$\left[ -m_1 \dddot{x} + m_1 g \sin \alpha - \frac{\Theta_A}{r^2} \dddot{x} + m_2 g - m_2 \dddot{x} \right] \delta x = 0.$$

Since $\delta x \neq 0$, the expression in the brackets must vanish. Thus,

$$\dddot{x} = \dddot{x}_2 = g \frac{m_2 - m_1 \sin \alpha}{m_1 + m_2 + \frac{\Theta_A}{r^2}}.$$ 

Note that $\dddot{x} < 0$ for $m_1 \sin \alpha > m_2$. In this case, the wheel rolls down the inclined plane.
Example 4.13 Two drums are connected by a rope and carry blocks of weights $m_1 g$ and $m_2 g$ (Fig. 4.14). Drum Φ is driven by the moment $M_0$. Determine the acceleration of block ② using d’Alembert’s principle. Neglect the mass of the ropes.

Solution We introduce the inertial forces $m_i \ddot{x}_i$ and the pseudo moments $\Theta_A \ddot{\varphi}_1$, $\Theta_B \ddot{\varphi}_2$. They act in the directions opposite to the chosen positive coordinate directions. D’Alembert’s principle requires

$$\delta U + \delta U_I = 0,$$

which leads to

$$-m_1(g + \ddot{x}_1)\delta x_1 + m_2(g - \ddot{x}_2)\delta x_2 + M_0\delta \varphi_1 - \Theta_A \ddot{\varphi}_1 \delta \varphi_1 - \Theta_B \ddot{\varphi}_2 \delta \varphi_2 = 0.$$

With the kinematic relations

$$\begin{align*}
x_1 &= r_1 \varphi_1, \\
x_2 &= r_2 \varphi_2, \\
\varphi_1 &= \varphi_2, \\
\delta \varphi_1 &= \delta \varphi_2 = \frac{\delta x_2}{r_2}, \\
\delta x_1 &= \frac{r_1}{r_2} \delta x_2,
\end{align*}$$

$$\dot{x}_1 = \frac{r_1}{r_2} x_2,$$

$$\dot{x}_2 = \frac{r_2}{r_1} x_1.$$

Fig. 4.14
we obtain

\[
\begin{align*}
\{ & -m_1 \left(g + \frac{r_1}{r_2} \ddot{x}_2\right) \frac{r_1}{r_2} + m_2(g - \ddot{x}_2) \\
& + \frac{M_0}{r_2} - \frac{\Theta_A}{r_2^2} \ddot{x}_2 - \frac{\Theta_B}{r_2^2} \ddot{x}_2 \} \delta x_2 = 0.
\end{align*}
\]

Since \( \delta x_2 \neq 0 \), the expression in the curly brackets must vanish. Thus, the acceleration of block \( \ddot{x}_2 \) is

\[
\ddot{x}_2 = g \frac{1 - \frac{m_1 r_1}{m_2 r_2} + \frac{M_0}{r_2 m_2 g}}{1 + \frac{m_1}{m_2} \left(\frac{r_1}{r_2}\right)^2 + \frac{\Theta_A}{m_2 r_2^2} + \frac{\Theta_B}{m_2 r_2^2}}.
\]
Example 4.14 The system shown in Fig. 4.15 consists of a block (mass $M$), a homogeneous disk (mass $m$, radius $r$) and two springs (spring constant $k$). The block moves on a frictionless surface; the disk rolls without slipping on the block. A force $F(t)$ acts on the block. Derive the equations of motion using Lagrange’s formalism.

Solution The system has two degrees of freedom. We choose the displacement $x$ of the block and the angle of rotation $\varphi$ of the disk to be the generalized coordinates.

The two springs are unstressed for $x = 0$ and $\varphi = 0$. The kinetic energy and the potential energy of the springs, respectively, are

$$T = M\dot{x}^2/2 + \Theta_C\dot{\varphi}^2/2 + mr^2\dot{\varphi}^2/2,$$

$$V = kx^2/2 + kr^2\varphi^2/2.$$

With $\Theta_C = mr^2/2$ and the kinematic relation $v_C = \dot{x} - \dot{\varphi}$ we obtain the Lagrangian

$$L = T - V,$$

$$\rightarrow L = M\dot{x}^2/2 + mr^2\dot{\varphi}^2/4 + m(\dot{x} - r\dot{\varphi})^2/2 - kx^2/2 - kr^2\varphi^2/2.$$

Since the force $F(t)$ is not given from a potential, we have to apply the Lagrangean equations in the form

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = Q_x, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} = Q_{\varphi}.$$

The generalized forces $Q_x$ and $Q_{\varphi}$ follow from the virtual work $\delta U$ of the force $F(t)$:

$$\delta U = Q_x \delta x + Q_{\varphi} \delta \varphi = F(t)\delta x \quad \rightarrow \quad Q_x = F(t), \quad Q_{\varphi} = 0.$$
To set up the equations of motion, the following derivatives have to be calculated:

\[
\frac{\partial L}{\partial \dot{x}} = M\ddot{x} + m(\dot{x} - r\dot{\phi}) ,
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = M\ddot{x} + m(\dot{x} - r\dot{\phi}) ,
\]

\[
\frac{\partial L}{\partial \dot{\phi}} = mr^2\ddot{\phi}/2 - mr(\dot{x} - r\dot{\phi}) ,
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = mr^2\dddot{\phi}/2 - mr(\ddot{x} - r\ddot{\phi}) ,
\]

\[
\frac{\partial L}{\partial x} = -kx , \quad \frac{\partial L}{\partial \phi} = -kr^2\phi .
\]

Thus, we obtain

\[
(M + m)\ddot{x} - mr\dddot{\phi} + kx = F(t) ,
\]

\[
-m\dddot{x} + \frac{3}{2}mr\dddot{\phi} + kr\dot{\phi} = 0 .
\]
Example 4.15  Fig. 4.16 shows two blocks of mass $m_1$ and $m_2$ which can glide on a frictionless surface. They are coupled by springs (stiffnesses $k_1$, $k_2$, $k_3$).

Derive the equations of motion using the Lagrange formalism.

Solution  The system is conservative; it has two degrees of freedom. We introduce the two coordinates $x_1$ and $x_2$ which describe the positions of the two blocks. They are measured from the equilibrium positions of the blocks. The kinetic and the potential energy, respectively, are given by

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2,$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k_3(x_2 - x_1)^2.$$

Thus, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k_3(x_2 - x_1)^2.$$

To set up the Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = 0,$$

the following derivatives must be calculated:

$$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_1} \right) = m_1\ddot{x}_1, \quad \frac{\partial L}{\partial x_1} = -k_1x_1 + k_3(x_2 - x_1),$$

$$\frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_2} \right) = m_2\ddot{x}_2, \quad \frac{\partial L}{\partial x_2} = -k_2x_2 - k_3(x_2 - x_1).$$
Hence, we obtain

\[ m_1 \ddot{x}_1 + k_1 x_1 - k_3 (x_2 - x_1) = 0 \]
\[ \rightarrow \quad m_1 \ddot{x}_1 + (k_1 + k_3) x_1 - k_3 x_2 = 0, \]

\[ m_2 \ddot{x}_2 + k_2 x_2 + k_3 (x_2 - x_1) = 0 \]
\[ \rightarrow \quad m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_3 x_1 = 0. \]

Note: The two coupled differential equations describe the coupled free vibrations of the two blocks. In the special case of \( k_3 = 0 \) the system is decoupled and we obtain two independent equations of motion for two systems, each with one degree of freedom.
Example 4.16  Two simple pendulums (each mass \( m \), length \( l \)) are connected by a spring (spring constant \( k \), unstretched length \( b \)) as shown in Fig. 4.17.

Derive the equations of motion using the Lagrange formalism.

Solution  The system has two degrees of freedom. We choose the generalized coordinates \( \varphi_1 \) and \( \varphi_2 \) as shown in the figure.

With the kinetic and the potential energy, respectively,

\[
T = ml^2(\dot{\varphi}_1 - \dot{\varphi}_2)^2/2 + ml^2(\dot{\varphi}_1 + \dot{\varphi}_2)^2/2
\]

\[
\rightarrow T = ml^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2)
\]

\[
V = -mgl \cos(\varphi_1 - \varphi_2) - mgl \cos(\varphi_1 + \varphi_2) + k(2l \sin \varphi_2 - b)^2/2
\]

\[
\rightarrow V = -2mgl \cos \varphi_1 \cos \varphi_2 + k(2l \sin \varphi_2 - b)^2/2
\]

(zero level: point 0 and unstressed spring) the Lagrangian becomes

\[
L = T - V
\]

\[
\rightarrow L = ml^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) + 2mgl \cos \varphi_1 \cos \varphi_2 - k(2l \sin \varphi_2 - b)^2/2.
\]

To set up the Lagrange equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}_1} \right) - \frac{\partial L}{\partial \varphi_1} = 0, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}_2} \right) - \frac{\partial L}{\partial \varphi_2} = 0
\]
the following derivatives are needed:

\[
\begin{align*}
\frac{\partial L}{\partial \dot{\varphi}_1} &= 2ml^2 \dot{\varphi}_1, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}_1} \right) &= 2ml^2 \ddot{\varphi}_1, \\
\frac{\partial L}{\partial \varphi_1} &= -2mgl \sin \varphi_1 \cos \varphi_2, \\
\frac{\partial L}{\partial \varphi_2} &= 2ml^2 \dot{\varphi}_2, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}_2} \right) &= 2ml^2 \ddot{\varphi}_2, \\
\frac{\partial L}{\partial \varphi_2} &= -2mgl \cos \varphi_1 \sin \varphi_2 - k(2l \sin \varphi_2 - b)l \cos \varphi_2.
\end{align*}
\]

This leads to the equations of motion:

\[
\begin{align*}
l \ddot{\varphi}_1 + g \sin \varphi_1 \cos \varphi_2 &= 0, \\
ml \ddot{\varphi}_2 + mg \cos \varphi_1 \sin \varphi_2 + k(2l \sin \varphi_2 - b)l \cos \varphi_2 &= 0.
\end{align*}
\]
Example 4.17  A disk (weight $m_2g$, moment of inertia $\Theta_2$) glides along a frictionless homogeneous bar of weight $m_1g$ (Fig. 4.18).

Find the equations of motion using the Lagrange formalism.

Solution  The system is conservative; it has two degrees of freedom. Its position is uniquely determined by the distance $x$ of point $C_1$ from pin 0 and by the angle $\varphi$ (generalized coordinates). With the kinetic and the potential energy, respectively,

$$T = \frac{1}{2}\left(\frac{m_1l^2}{3}\right)\dot{\varphi}^2 + \left\{\frac{1}{2}m_2\left(x\dot{\varphi}^2 + \dot{x}^2\right) + \frac{1}{2}\Theta_2\dot{\varphi}^2\right\},$$

$$V = -m_1\frac{l}{2} \cos \varphi - m_2x \cos \varphi = -\left[m_1\frac{l}{2} + m_2x\right]g \cos \varphi$$

(zero level at 0) and the Lagrangian $L = T - V$, we can write down the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

To this end, the following derivatives are needed:

$$\frac{\partial L}{\partial \dot{\varphi}} = \left(\frac{m_1l^2}{3} + m_2x^2 + \Theta_2\right)\dot{\varphi}, \quad \frac{\partial L}{\partial \dot{x}} = -\left(m_1\frac{l}{2} + m_2x\right)g \sin \varphi,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \left(m_1\frac{l^2}{3} + m_2x^2 + \Theta_2\right)\ddot{\varphi} + 2m_2x\dot{x} \dot{\varphi}, \quad \frac{\partial L}{\partial x} = m_2\ddot{x},$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_2\ddot{\varphi}, \quad \frac{\partial L}{\partial x} = m_2x\dot{\varphi}^2 + m_2g \cos \varphi.$$

This leads to the coupled equations of motion:

$$\left(m_1\frac{l^2}{3} + m_2x^2 + \Theta_2\right)\ddot{\varphi} + 2m_2x\dot{x} \dot{\varphi} + \left(m_1\frac{l}{2} + m_2x\right)g \sin \varphi = 0,$$

$$m_2\ddot{x} - m_2x\dot{\varphi}^2 - m_2g \cos \varphi = 0 \quad \rightarrow \quad \ddot{x} - x\dot{\varphi}^2 - g \cos \varphi = 0.$$
Example 4.18 A thin half-cylindrical shell of weight \( W = mg \) rolls without sliding on a flat surface (Fig. 4.19).

Derive the equation of motion using the Lagrange formalism.

Solution The system is conservative; it has one degree of freedom. We introduce the coordinate \( \varphi \) as shown in the figure. With the distance \( a = 2r/\pi \) of the center of mass \( C \) from the center \( M \) of the shell we have

\[
\Theta_C = \Theta_M - a^2 m = r^2 m - a^2 m = (1 - 4/\pi^2)mr^2,
\]

\[
x_C = r \varphi - a \sin \varphi \quad \rightarrow \quad \dot{x}_C = r \dot{\varphi} - a \dot{\varphi} \cos \varphi,
\]

\[
y_C = a \cos \varphi \quad \rightarrow \quad \dot{y}_C = -a \dot{\varphi} \sin \varphi.
\]

Thus, the potential energy \( V \), the kinetic energy \( T \), the Lagrangian and the pertinent derivatives are

\[
V = mga(1 - \cos \varphi) = \frac{2}{\pi} mrg(1 - \cos \varphi)
\]

\[
T = \frac{1}{2} m (\dot{x}_C^2 + \dot{y}_C^2) = \frac{1}{2} m r^2 \dot{\varphi}^2 \left[(1 - \frac{2}{\pi} \cos \varphi)^2 + \left(\frac{2}{\pi} \sin \varphi\right)^2 \left(1 - \frac{4}{\pi^2}\right) - 2 \frac{2}{\pi} g (1 - \cos \varphi)\right],
\]

\[
L = T - V = m r \dot{\varphi} \left[r^2 \left(1 - \frac{2}{\pi} \cos \varphi\right) - 2 \frac{2}{\pi} g (1 - \cos \varphi)\right],
\]

\[
\frac{\partial L}{\partial \dot{\varphi}} = m r \left[2 \dot{\varphi} \left(1 - \frac{2}{\pi} \cos \varphi\right)\right],
\]

\[
\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}}\right) = m r \left[2 \dot{\varphi} \left(1 - \frac{2}{\pi} \cos \varphi\right) + \frac{4}{\pi} r^2 \dot{\varphi}^2 \sin \varphi\right],
\]

\[
\frac{\partial L}{\partial \varphi} = m r \left[2 \frac{2}{\pi} \dot{\varphi}^2 \sin \varphi - \frac{2}{\pi} g \sin \varphi\right].
\]
Introduction into
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} = 0 \]
yields the equation of motion:
\[ \ddot{\varphi} (\pi - 2 \cos \varphi) + \dot{\varphi}^2 \sin \varphi + \frac{g}{r} \sin \varphi = 0 . \]
Example 4.19  A block (mass $m_1$) can move horizontally on a smooth surface (Fig. 4.20). A simple pendulum (mass $m_2$) is connected to the block by a pin.

Find the equations of motion using the Lagrange formalism.

Solution  The system is conservative; it has two degrees of freedom. We use the generalized coordinates $x$ and $\varphi$ as shown in the figure. With the zero-level of the potential of the force $m_2g$ chosen at the height of the mass $m_1$, we have

$$V = -m_2gl \cos \varphi,$$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[ (\dot{x} + l \dot{\varphi} \cos \varphi)^2 + (l \dot{\varphi} \sin \varphi)^2 \right],$$

$$L = T - V = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + m_2 l \dot{\varphi} \cos \varphi + \frac{1}{2} m_2 \dot{\varphi}^2 + m_2 g l \cos \varphi.$$  

Introduction of the derivatives

$$\frac{\partial L}{\partial \dot{\varphi}} = m_2 \dot{x} \cos \varphi + m_2 \ddot{\varphi}, \quad \frac{\partial L}{\partial \varphi} = -m_2 \dot{x} \dot{\varphi} \sin \varphi - m_2 gl \sin \varphi,$$

$$d \left( \frac{\partial L}{\partial \dot{x}} \right) = m_2 \ddot{x} \cos \varphi - m_2 \dot{x} \ddot{\varphi} \sin \varphi + m_2 l \ddot{\varphi},$$

$$\frac{\partial L}{\partial x} = (m_1 + m_2) \ddot{x} + m_2 l \ddot{\varphi} \cos \varphi, \quad \frac{\partial L}{\partial \dot{x}} = 0,$$

$$d \left( \frac{\partial L}{\partial \dot{x}} \right) = (m_1 + m_2) \ddot{x} + m_2 l \ddot{\varphi} \cos \varphi - m_2 l \ddot{\varphi}^2 \sin \varphi.$$
into the Lagrange equations
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} = 0, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial x} \right) - \frac{\partial L}{\partial x} = 0
\]
yields the coupled equations of motion
\[
\ddot{x} \cos \varphi + l \ddot{\varphi} + g \sin \varphi = 0, \\
(m_1 + m_2) \ddot{x} + m_2 l \ddot{\varphi} \cos \varphi - m_2 l \dot{\varphi}^2 \sin \varphi = 0.
\]
### Example 5.11

The system in Fig 5.30 consists of three bars and a beam (with negligible masses) and a block (mass $m$).

Determine the circular frequency of the free vertical vibrations.

#### Solution

The system consisting of the truss and the beam is equivalent to a system consisting of two springs in parallel (both springs undergo the same elongation when the block is displaced). To determine the spring constant $k_B$ of the beam, we subject the beam to the force $F_B = 1$ which acts at the location of the block. This force produces the deflection (see Engineering Mechanics 2: Mechanics of Materials, Table 4.3)

$$w_B = \frac{1 \cdot (2l)^3}{48EI}.$$  

Thus, the spring constant is given by

$$k_B = \frac{1}{w_B} = \frac{48EI}{(2l)^3} = \frac{6EI}{l^3}.$$  

In order to find the spring constant $k_T$ of the truss, we apply the force $F_T = 1$ at bar $1$. This force causes the displacement (see Engineering Mechanics 2: Mechanics of Materials, Section 6.3)

$$w_T = \sum \frac{S_{1i}}{EA}.$$  

---

**Fig. 5.30**
With
\[ S_1 = 1, \quad S_2 = S_3 = \frac{1}{2} \sqrt{2}, \quad l_1 = l, \quad l_2 = l_3 = \sqrt{2} l \]
we obtain
\[
w_T = \frac{1}{EA} \left[ l^2 \cdot l + 2 \cdot \left( \frac{\sqrt{2}}{2} \right)^2 \sqrt{2} l \right] = (1 + \sqrt{2}) \frac{l}{EA} \]

\[ \Rightarrow k_T = \frac{1}{w_T} = \frac{EA}{(1 + \sqrt{2})l}. \]

Now, we replace the two springs in parallel by an equivalent single spring. Its spring constant \( k^* \) is given by
\[ k^* = k_B + k_T = \frac{6 EI}{l^3} + \frac{EA}{(1 + \sqrt{2})l}. \]

Thus, the eigenfrequency is
\[
\omega = \sqrt{\frac{k^*}{m}} = \frac{1}{l} \sqrt{\frac{1}{m} \left( 6 EI + \frac{EA l^2}{1 + \sqrt{2}} \right)}.\]
Example 5.12 The system in Fig. 5.31 consists of a homogeneous drum (mass $M$, radius $r$), a block (mass $m$), a spring (spring constant $k$) and a string (with negligible mass). The support of the drum is frictionless. Assume that there is no slip between the string and the drum.

Determine the natural frequency of the system.

Solution We first draw the free-body diagram. The position of the block is given by the coordinate $x$, measured from the position with an unstretched spring. Next, we write down the equations of motion for the block

\[ m \ddot{x} = mg - S_1 \]

and for the drum

\[ B: \quad M r^2 \ddot{\phi}/2 = S_1 r - S_2 r. \]

In addition, we need the kinematic relation

\[ \dot{x} = r \dot{\phi} \]

and the equation

\[ S_2 = kx \]

for the restoring force in the spring. Solving yields the differential equation for harmonic oscillations:

\[ \ddot{x} + \frac{2k}{M + 2m} x = \frac{2mg}{M + 2m}. \]

Thus, the natural frequency is

\[ \omega = \sqrt{\frac{2k}{M + 2m}}. \]
Example 5.13  Two drums rotate in opposite directions as shown in Fig. 5.32. They support a homogeneous board of weight $W$ (mass $m$). The coefficient of kinetic friction between the drums and the board is $\mu$.

Show that the board undergoes a harmonic vibration and determine the natural frequency.

Solution  We separate the various parts of the system. The free-body diagram shows the forces acting if the board is displaced by an amount $x$ (the friction forces act on the drums in the opposite directions of the rotations).

Thus, the equation of motion in the $x$-direction of the board is given by

$$m \ddot{x} = R_2 - R_1 .$$

The normal forces follow from force equilibrium in the vertical direction and from moment equilibrium:

$$N_1 = W \frac{a}{2} + x$$
$$N_2 = W \frac{a}{2} - x .$$

With the law of kinetic friction $R = \mu N$, we obtain

$$R_2 - R_1 = \mu (N_2 - N_1) = -\mu mg \frac{2x}{a} .$$
The equation of motion

\[ m\ddot{x} = -\mu mg \frac{2x}{a} \]

thus leads to the differential equation for harmonic vibrations:

\[ \ddot{x} + 2\mu \frac{g}{a} x = 0. \]

The natural frequency is given by

\[ \omega = \sqrt{2\mu \frac{g}{a}}. \]

Note that the natural frequency does not depend on the angular velocity \( \Omega \) of the drums.
Example 5.14 A homogeneous bar (weight \( W = mg \), length \( l \)) is submerged in a viscous fluid and undergoes vibrations about point \( A \) (Fig. 5.33). The drag force \( F_d \) acting on every point of the bar is proportional to the local velocity (proportionality factor \( \beta \)).

a) Derive the equation of motion. Assume small amplitudes and neglect the buoyancy. b) Calculate the value \( \beta = \beta^* \) for critical damping.

Solution a) We consider an arbitrary element of length \( dx \) of the bar. It is subjected to the drag force

\[
dF_d = \beta v(x) \, dx = \beta x \dot{\varphi} \, dx.
\]

We restrict ourselves to small amplitudes (\( \sin \varphi \approx \varphi \)). In this case the principle of angular momentum yields

\[
\begin{align*}
\ddot{\varphi} + \beta l \frac{m}{2} \dot{\varphi} + 3g \frac{m}{2l} \varphi & = 0 \\
\ddot{\varphi} + 2\xi \dot{\varphi} + \omega^2 \varphi & = 0
\end{align*}
\]

where

\[
\xi = \frac{\beta l}{2m} , \quad \omega^2 = \frac{3g}{2l}.
\]
b) Critical damping is characterized by

\[ \xi = \omega \quad \text{or} \quad \zeta = 1. \]

Thus,

\[ \frac{\beta^* l}{2m} = \sqrt{\frac{3\ g}{2\ l}} \quad \rightarrow \quad \beta^* = \frac{m}{T} \sqrt{\frac{6\ g}{l}}. \]
Example 5.15 The pendulum of a clock consists of a homogeneous rod (mass \( m \), length \( l \)) and a homogeneous disk (mass \( M \), radius \( r \)) whose center is located at a distance \( a \) from point \( A \) (Fig. 5.34). Assume small amplitudes and determine the natural frequency of the corresponding oscillations. Choose \( m = M \) and \( r \ll a \) and calculate the ratio \( a/l \) which yields the maximum eigenfrequency.

Solution We introduce the angle \( \phi \) as shown and apply the principle of angular momentum:

\[
\dot{\Theta}_A \dot{\phi} = -\frac{mgl}{2} \sin \phi - Mg a \sin \phi
\]

where

\[
\Theta_A = \frac{ml^2}{3} + M \left( r^2 + 2a^2 \right)/2.
\]

If we assume small amplitudes (\( \sin \phi \approx \phi \)), we can linearize the equation of motion:

\[
\ddot{\phi} + \frac{(ml + 2Ma)g}{2\Theta_A} \phi = 0.
\]

Hence, the eigenfrequency is obtained as

\[
\omega = \sqrt{\frac{(ml + 2Ma)g}{2\Theta_A}}.
\]

In the special case of \( m = M \) and \( r \ll a \) the natural frequency simplifies to

\[
\omega = \sqrt{\frac{3l + 6a}{2l^2 + 6a^2} g}.
\]

The ratio \( a/l \) which yields the maximum eigenfrequency is found by setting the derivative \( d\omega/da \) equal to zero:

\[
\frac{d\omega}{da} = 0 \quad \Rightarrow \quad a/l = \frac{1}{2} \left( \sqrt{\frac{7}{3}} - 1 \right).
\]
Example 5.16 The system in Fig 5.35 consists of a homogeneous pulley (mass $M$, radius $r$), a block (mass $m$) and a spring (spring constant $k$).

Determine the equation of motion for the block and its solution for the initial conditions $x(0) = 0$, $v(0) = v_0$. Neglect the mass of the string and any lateral motion.

Solution We separate the pulley and the block and measure the displacements $x$ of the block and $x_A$ of point $A$ from the position of equilibrium. With this choice, we do not have to consider the weights $Mg$ and $mg$ in the free-body diagram. Thus, the equations of motion are

1. $m\ddot{x} = -S_1$,
2. $M\ddot{x}_A = S_1 + S_2 - kx_A$,
3. $\Theta_A \ddot{\varphi} = rS_1 - rS_2$.

If we use the kinematic relations (II\(\hat{\jmath}\): instantaneous center of rotation, see the figure)

$$x_A = \frac{x}{2}, \quad 2r\varphi = \varphi \rightarrow \ddot{x}_A = \frac{\ddot{x}}{2}, \quad \ddot{\varphi} = \frac{\ddot{x}}{2r}$$

and $\Theta_A = Mr^2/2$ we can solve the equations of motion for $x$ and obtain

$$\ddot{x} + \frac{k}{4m + \frac{3}{2}M} x = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{4m + \frac{3}{2}M}}.$$
The general solution of this differential equation is given by
\[ x(t) = A \cos \omega t + B \sin \omega t. \]
The initial conditions
\[ x(0) = 0 \rightarrow A = 0, \quad \dot{x}(0) = v_0 \rightarrow B = \frac{v_0}{\omega} \]
lead to
\[ x(t) = \frac{v_0}{\omega} \sin \omega t. \]
Example 5.17  A wheel (mass $m$, radius $r$) rolls without slipping on a circular path (Fig. 5.36). The mass of the rod (length $l$) can be neglected; the joints are frictionless.

Derive the equation of motion and determine the natural frequency of small oscillations.

Solution  We apply conservation of energy

$$T + V = \text{const}$$

to derive the equation of motion. The kinetic energy of the rolling wheel is given by (see the figure)

$$T = m v_c^2/2 + \Theta C \dot{\psi}^2/2,$$

the potential energy is (zero-level at the height of point $A$)

$$V = -mgl \cos \phi.$$  

With the mass moment of inertia $\Theta_C = mr^2/2$ and the kinematic relations

$$v_c = l \dot{\phi}, \quad v_c = r \dot{\psi},$$

the kinetic energy can be written as

$$T = 3ml^2 \dot{\phi}^2/4.$$  

Introduction into the expression for conservation of energy gives

$$3l^2 \dot{\phi}^2/4 - g \cos \phi = \text{const}.$$  

Differentiation yields

$$\frac{3}{2} \dot{\phi} \ddot{\phi} + g \dot{\phi} \sin \phi = 0 \quad \rightarrow \quad \ddot{\phi} + \frac{2g}{3l} \sin \phi = 0.$$
If we restrict ourselves to small amplitudes ($\sin \varphi \approx \varphi$) this equation reduces to

$$\ddot{\varphi} + \frac{2g}{3l} \varphi = 0.$$  

Thus, the natural frequency is obtained as

$$\omega = \sqrt{\frac{2g}{3l}}.$$
Example 5.18 The simple pendulum in Fig. 5.37 is attached to a spring (spring constant $k$) and a dashpot (damping coefficient $d$).

a) Determine the maximum value of the damping coefficient $d$ so that the system undergoes vibrations. Assume small amplitudes.

b) Find the damping ratio $\zeta$ so that the amplitude is reduced to $1/10$ of its initial value after 10 full cycles. Calculate the corresponding period $T_d$.

Solution

a) The equation of motion follows from the principle of angular momentum (rotation about point $A$; small amplitudes: $\sin \varphi \approx \varphi$, $\cos \varphi \approx 1$):

$$\Theta_A \ddot{\varphi} = -F_d a - F_k 2a - mg a \varphi.$$

With

$$\Theta_A = m (2a)^2, \quad F_d = da \dot{\varphi}, \quad F_k = k(2a) \varphi,$$

we obtain

$$\ddot{\varphi} + \frac{d}{4m} \dot{\varphi} + \left( \frac{k}{m} + \frac{g}{2a} \right) \varphi = 0 \quad \rightarrow \quad \ddot{\varphi} + 2\zeta \dot{\varphi} + \omega^2 \varphi = 0,$$

where

$$\zeta = \frac{d}{8m}, \quad \omega^2 = \frac{k}{m} + \frac{g}{2a}.$$
In order to have oscillations, the system must be underdamped:

\[ \zeta < 1 \quad \rightarrow \quad \xi < \omega \quad \rightarrow \quad \frac{d}{8m} < \sqrt{\frac{k}{m} + \frac{g}{2a}} \]

\[ \rightarrow \quad d < 8\sqrt{\frac{km + gm^2}{2a}}. \]

b) The necessary damping ratio follows with \( x_{n+10} = x_n/10 \) from the logarithmic decrement:

\[ 10 \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} = \ln \frac{x_n}{x_{n+10}} = \ln 10 \]

\[ \rightarrow \quad \zeta = \sqrt{\frac{20\pi^2}{(\ln 10)^2} + 1} = 0.037. \]

This leads to the period

\[ T_d = \frac{2\pi}{\omega \sqrt{1 - \zeta^2}} \approx \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{2am}{2ak + gm}}. \]
Example 5.19 The structure in Fig. 5.38 consists of an elastic beam (flexural rigidity $EI$, axial rigidity $EA \to \infty$, negligible mass) and three rigid bars (with negligible masses). The block (mass $m$) is suspended from a spring (spring constant $k$).

Determine the eigenfrequency of the vertical oscillations of the block.

Solution We reduce the structure to an equivalent system of a spring (spring constant $k^*$, see the figure) and a mass. To this end we first replace the beam and the bars by a spring with the spring constant $k_B$. We can determine $k_B$ if we subject the beam to a force $F$ which acts at the free end. This force produces the deflection

$$w = \frac{Fa^3}{EI}$$

(see Engineering Mechanics 2: Mechanics of Materials, Example 6.22). The spring constant is obtained from

$$k_B = \frac{F}{w} \quad \Rightarrow \quad k_B = \frac{EI}{a^3}.$$ 

The displacement of the block is the sum of the elongations of the springs with spring constants $k$ and $k_B$. Therefore, the beam/bars and the given spring act as springs in series. Thus, the spring constant $k^*$ of the equivalent system follows from

$$\frac{1}{k^*} = \frac{1}{k_B} + \frac{1}{k} \quad \Rightarrow \quad k^* = \frac{kEI}{ka^3 + EI}.$$ 

This yields the eigenfrequency

$$\omega = \sqrt{\frac{k^*}{m}} \quad \Rightarrow \quad \omega = \sqrt{\frac{kEI}{(ka^3 + EI)m}}.$$
**Example 5.20** A rod (length \( l \), with negligible mass) is elastically supported at point \( A \) (Fig. 5.39). The rotational spring (spring constant \( k_T \)) is unstretched for \( \varphi = 0 \). The rod carries a point mass \( m \) at its free end.

Derive the equation of motion. Determine the spring constant so that \( \varphi = \pi / 6 \) is an equilibrium position. Calculate the natural frequency of small oscillations about this equilibrium position.

**Solution** To derive the equation of motion we apply the principle of angular momentum:

\[
\hat{A}: \Theta_A \ddot{\varphi} = mg l \cos \varphi - M_A.
\]

With the mass moment of inertia \( \Theta_A = ml^2 \) and the restoring moment \( M_A = k_T \varphi \) we obtain

\[
\ddot{\varphi} = \frac{g}{l} \cos \varphi - \frac{k_T}{ml^2} \varphi .
\]

Since \( \varphi = \varphi_0 = \pi / 6 \) is a position of equilibrium, the condition \( \ddot{\varphi}(\pi / 6) = 0 \) leads to the required spring constant (note that \( \cos(\pi / 6) = \sqrt{3} / 2 \)):

\[
\frac{\sqrt{3} g}{2 l} - \frac{\pi k_T}{6 ml^2} \varphi_0 = 0 \quad \rightarrow \quad k_T = \frac{3 \sqrt{3}}{\pi} mg l .
\]

Now we consider small oscillations about this position of equilibrium. We assume that

\[
\varphi = \varphi_0 + \psi \quad \text{with} \quad |\psi| \ll 1 .
\]

Introduction into the equation of motion yields

\[
\ddot{\psi} = \frac{g}{l} \cos(\varphi_0 + \psi) - \frac{k_T}{ml^2} (\varphi_0 + \psi) .
\]

We use the trigonometric relation

\[
\cos(\varphi_0 + \psi) = \cos \varphi_0 \cos \psi - \sin \varphi_0 \sin \psi,
\]

\[
\rightarrow \quad \cos(\varphi_0 + \psi) = \frac{\sqrt{3}}{2} \cos \psi - \frac{1}{2} \sin \psi .
\]
and linearize \((\cos \psi \approx 1, \sin \psi \approx \psi)\) to obtain
\[
\ddot{\psi} = -\frac{1}{2}g \psi - \frac{k_T}{ml^2} \psi + \frac{\sqrt{3} g}{2l} \psi - \frac{\pi}{6} \frac{k_T}{ml^2}
\]
\[
\rightarrow \ddot{\psi} + \left(\frac{1}{2} + \frac{3\sqrt{3}}{\pi}\right) \frac{g}{l} \psi = 0.
\]
Thus, the natural frequency is given by
\[
\omega = \sqrt{\left(\frac{1}{2} + \frac{3\sqrt{3}}{\pi}\right) \frac{g}{l}}.
\]
Example 5.21 A single story frame consists of two rigid columns (with negligible masses), a rigid beam of mass \( m \) and a spring-dashpot system as shown in Fig. 5.40. The ground is forced to vibration by an earthquake; the acceleration \( \ddot{u}_E = b_0 \cos \Omega t \) is known from measurements.

Determine the maximum amplitude of the steady state vibrations. Assume that the system is underdamped and that the vibrations have small amplitudes.

Solution We assume that the amplitudes of the vibrations are small. Then the elongation of the diagonal is obtained as

\[
\Delta = \frac{\sqrt{2}}{2} (x - u_E).
\]

The elongation produces the force

\[
F = k \Delta + d \dot{\Delta}
\]

in the spring-dashpot system (see the figure). The equation of motion of the beam is given by

\[
\Rightarrow: \quad m \ddot{x} = F \frac{\sqrt{2}}{2}
\]

\[
\Rightarrow \quad m \ddot{x} + \frac{d}{2} (x - \dot{u}_E) + \frac{k}{2} (x - u_E) = 0.
\]

Thus, the relative displacement \( y = x - u_E \) is described by

\[
m \ddot{y} + \frac{d}{2} \dot{y} + \frac{k}{2} y = m b_0 \cos \Omega t
\]

\[
\Rightarrow \quad \frac{1}{\omega^2} \ddot{y} + \frac{2 \zeta}{\omega} \dot{y} + y = y_0 \cos \Omega t,
\]

where

\[
\omega^2 = \frac{k}{2m}, \quad \zeta = \frac{d}{2} \sqrt{\frac{1}{2km}}, \quad y_0 = \frac{2mb_0}{k}.
\]
The maximum amplitude $A$ is obtained for $\eta = \Omega/\omega \approx 1$ (resonance!). In the case of small damping ($\zeta \ll 1$) we obtain

$$A = y_0 V_{\text{max}} \approx \frac{y_0}{2\zeta} = 2\sqrt{\frac{\nu_0}{d}} \sqrt{\frac{m^3}{k}}.$$
Example 5.22 The undamped system in Fig. 5.41 consists of a block (mass $m = 4 \text{ kg}$) and a spring (spring constant $k = 1 \text{ N/m}$). The block is subjected to a force $F(t)$. The initial conditions $x(0) = x_0 = 1 \text{ m}$, $\dot{x}(0) = 0$ and the response

$$x(t) = x_0 \left[ \cos \frac{t}{2t_0} + 20 \left( 1 - \cos \frac{t - T}{2t_0} \right) \langle t - T \rangle \right]$$

to the excitation are given. Here, $t_0 = 1 \text{ s}$, $T = 5 \text{ s}$ and $\langle t - T \rangle = 0$ for $t < T$ and $\langle t - T \rangle = 1$ for $t > T$.

Calculate the force $F(t)$.

Solution First we calculate the force $F$ for $t < T$. Then $\langle t - T \rangle = 0$ and the response is given by

$$x(t) = x_0 \cos \frac{t}{2t_0}.$$ 

The unknown force follows from the equation of motion:

$$F = m\ddot{x} + kx.$$ 

With

$$\dot{x} = -\frac{x_0}{2t_0} \sin \frac{t}{2t_0} \quad \text{and} \quad \ddot{x} = -\frac{x_0}{4t_0^2} \cos \frac{t}{2t_0}$$

we obtain

$$F(t) = \left( -\frac{mx_0}{4t_0^2} + kx_0 \right) \cos \frac{t}{2t_0}.$$ 

Inserting the given numerical values of the parameters leads to $F(t) \equiv 0$ for $t < T$. 

Fig. 5.41
Now we consider the case $t > T$. Then $(t - T)^0 = 1$ and the response follows as

$$x(t) = x_0 \left[ \cos \frac{t}{2t_0} + 20 \left( 1 - \cos \frac{t - T}{2t_0} \right) \right]$$

$$\Rightarrow \dot{x} = x_0 \left[ -\sin \frac{t}{2t_0} + 20 \sin \frac{t - T}{2t_0} \right]$$

$$\Rightarrow \ddot{x} = \frac{x_0}{4t_0^2} \left[ -\cos \frac{t}{2t_0} + 20 \cos \frac{t - T}{2t_0} \right].$$

Introduction into the equation of motion yields

$$F(t) = \left( \frac{mx_0}{4t_0^2} + kx_0 \right) \cos \frac{t}{2t_0}$$

$$+ \left( \frac{20mx_0}{4t_0^2} - 20kx_0 \right) \cos \frac{t - T}{2t_0} + 20kx_0$$

$$\Rightarrow F(t) \equiv 20 \text{ N for } t > T.$$
Example 5.23  A simplified model of a car (mass $m$) is given by a spring-mass system (Fig. 5.42). The car drives with constant velocity $v_0$ over an uneven surface in the form of a sine function (amplitude $U_0$, wavelength $L$).

a) Derive the equation of motion and determine the forcing frequency $\Omega$.

b) Find the amplitude of the vertical vibrations as a function of the velocity $v_0$.

c) Calculate the critical velocity $v_c$ (resonance!).

Solution  a) We denote the vertical displacement of the car by $x$, the uneven surface is described by $u$. Then Newton’s law reads

$$\ddot{x} = -k(x - u).$$

With the position of the car, $s = v_0 t$, in the horizontal direction we obtain

$$u = U_0 \cos \frac{2\pi s}{L} = U_0 \cos \frac{2\pi v_0 t}{L} = U_0 \cos \Omega t.$$

Thus,

$$m \ddot{x} + k x = k U_0 \cos \Omega t \quad \text{with} \quad \Omega = \frac{2\pi v_0}{L}.$$

b) We assume the solution of the equation of motion to be of the form of the right-hand side: $x = x_0 \cos \Omega t$. This leads to the amplitude of the steady state vibrations:

$$x_0 = \frac{U_0}{\sqrt{1 - \frac{4\pi^2 v_0^2}{\omega^2} \frac{m}{k}}}.$$

where $\omega^2 = k/m$.

c) Resonance occurs for $\Omega = \omega$:

$$\frac{4\pi^2 v_0^2}{L^2} \frac{m}{k} = 1 \quad \Rightarrow \quad v_c = \frac{L}{2\pi} \sqrt{\frac{k}{m}}.$$
Example 5.24 A homogeneous wheel (mass $m$) is attached to a spring (spring constant $k$). The wheel rolls without slipping on a rough surface which moves according to the function $u = u_0 \cos \Omega t$ (Fig. 5.43).

a) Determine the amplitude of the steady state vibrations.

b) Calculate the coefficient $\mu_0$ of static friction which is necessary to prevent slipping.

Solution

The equations of motion for the wheel are given by

\begin{align*}
\uparrow & : \quad 0 = N - mg, \quad \text{(a)} \\
\rightarrow & : \quad m \ddot{x} = -kx + H, \quad \text{(b)} \\
\rightarrow\leftarrow & : \quad \Theta_C \ddot{\varphi} = -rH. \quad \text{(c)}
\end{align*}

With the kinematic relation

\[ x = u + r\varphi \]

and $\Theta_C = mr^2/2$ we obtain from (b) and (c) the differential equation for forced vibrations:

\[ \ddot{x} + \frac{2}{3} \frac{k}{m} x = -\frac{1}{2} u_0 \Omega^2 \cos \Omega t. \]

a) We assume the solution to be of the form of the right-hand side: $x = x_0 \cos \Omega t$. This leads to the amplitude of the steady state vibrations:

\[ |x_0| = \frac{u_0}{\sqrt{\frac{2}{3} \frac{k}{m} \Omega^2 - 1}}. \]

b) The condition $|H|_{\text{max}} \leq \mu_0 N$ for static friction and (a), (b) yield the required coefficient of static friction:

\[ \mu_0 \geq \frac{u_0 \Omega}{3g} \left| \frac{k}{m} \frac{\Omega^2}{g} - 1 \right| \]

\[ \rightarrow \quad \mu_0 \geq \frac{u_0 \Omega}{3g} \left( \frac{2}{3} \frac{k}{m} \frac{\Omega^2}{g} - 1 \right). \]
Example 5.25 A small homogeneous disk (mass $m$, radius $r$) is attached to a large homogeneous disk (mass $M$, radius $R$) as shown in Fig. 5.44. The torsion spring (spring constant $k_T$) is unstretched in the position shown.

Determine the eigenfrequency of the oscillations. Assume small amplitudes.

Solution We apply the principle of angular momentum to derive the equation of motion:

\[ \hat{\Theta}_A \ddot{\varphi} = -k_T \varphi - mga \sin \varphi. \]

In the case of small amplitudes ($\sin \varphi \approx \varphi$) this equation reduces to

\[ \ddot{\varphi} + \omega^2 \varphi = 0 \quad \text{with} \quad \omega^2 = \frac{k_T + mga}{\Theta_A}. \]

Inserting the mass moment of inertia

\[ \Theta_A = \frac{MR^2}{2} + \left[ \frac{m_r^2}{2} + ma^2 \right] \]

we can write the eigenfrequency in the form

\[ \omega = \sqrt{\frac{k_T + mga}{\frac{1}{2}MR^2 + \left( \frac{m_r^2}{2} + a^2 \right)}}. \]

Note that the problem can also be solved with the aid of the conservation of energy (we choose $V = 0$ for $\varphi = 0$):

\[ T + V = \text{const} \quad \Rightarrow \quad \frac{1}{2} \Theta_A \dot{\varphi}^2 + \frac{1}{2} k_T \varphi^2 + mga(1 - \cos \varphi) = \text{const}. \]

Differentiation yields

\[ \Theta_A \ddot{\varphi} + k_T \varphi + mga \sin \varphi \dot{\varphi} = 0. \]

With $\sin \varphi \approx \varphi$ and $\varphi \neq 0$ we obtain the same result as above.
Example 5.26 The systems ⃗ and ⃗ in Fig. 5.45 consist of two beams (negligible masses, flexural rigidity EI), a spring (spring constant k) and a box (mass m).

Determine the spring constants *k* of the equivalent springs for the two systems.

Solution We reduce both systems to the equivalent simple systems of a mass and a spring.

In system ⃗, the three “springs” are attached to the mass. Therefore, they undergo the same deflection when the box is displaced: they act as springs in parallel. The equivalent spring constant *k* is the sum of the individual spring constants:

\[ k^* = \sum k_i. \]

We obtain the spring constants *k*L and *k*R of the right and the left beam, respectively, if we subject the cantilevers to a force 1 at their free ends. The corresponding deflections are

\[ w = \frac{1}{3} \left( \frac{l^3}{EI} \right) \]

(see Engineering Mechanics 2: Mechanics of Materials, Section 4.5) which leads to...
Thus, \( k^* = k_L + k_R + k = \frac{27}{8} \frac{EI}{a^3} + k = \frac{27EI + 8ka^3}{8a^3} \).

Now we consider system ②. Here, the two beams act as springs in parallel with equivalent spring constant \( k \). This then acts in series with given spring (spring constant \( k \)). Hence,

\[
\begin{align*}
\bar{k} &= k_L + k_R = \frac{27}{8} \frac{EI}{a^3}, \\
\frac{1}{k^*} &= \frac{1}{k} + \frac{1}{\bar{k}} = \frac{8a^3}{27EI} + \frac{1}{k} \\
\Rightarrow k^* &= \frac{27EIk}{27EI + 8ka^3} = \frac{27EI}{8a^3 + 27EI/k}.
\end{align*}
\]

Note that the stiffness of system ② is smaller than the one of system ①. Therefore it vibrates with a smaller frequency.
E5.27 **Example 5.27** A homogeneous wheel (mass $m$, moment of inertia $\Theta_C$, radius $r$) rolls without slipping on a rough beam (mass $M$). The beam moves without friction on roller supports (Fig. 5.46).

Determine the natural frequency of the system.

**Solution** We separate the wheel and the beam and introduce the coordinates $x_1$, $x_2$ and $\varphi$ (see the figure). The coordinates are measured from the position of equilibrium. Then the equations of motion are

\[ \begin{align*}
\text{①} & \quad \rightarrow : \quad m \ddot{x}_1 = -kx_1 - H, \\
\text{②} & \quad \rightarrow : \quad M \ddot{x}_2 = H, \\
\end{align*} \]

If we use the kinematic relation

\[ x_1 = x_2 + r \varphi \]

and solve for $x_1$ we obtain

\[ \ddot{x}_1 + \frac{k}{m + \frac{M}{\Theta_C} + 2Mr^2/\Theta_C} x_1 = 0. \]

Thus, the natural frequency is given by

\[ \omega = \sqrt{\frac{M}{m + \frac{M}{\Theta_C} + 2Mr^2/\Theta_C}}. \]
Chapter 6

Non-Inertial Reference Frames
Example 6.5  Point A of the simple pendulum (mass $m$, length $l$) in Fig. 6.8 moves with a constant acceleration $a_0$ to the right. Derive the equation of motion.

Solution  We introduce the $\xi, \eta$-coordinate system as shown in the figure. It is a translating coordinate system with point $A$ as the origin. The equation of motion in the moving system is

$$ma_r = F + F_f.$$  

The (real) force $F$ acting at the mass is given by

$$F = -S \sin \varphi e_\xi + (S \cos \varphi - mg) e_\eta,$$

and the fictitious force $F_f$ is

$$F_f = -ma_f = -ma_0 e_\xi.$$  

Note that the Coriolis force is zero since $\omega = 0$.

The components of the relative acceleration $a_r$ follow from the coordinates of the point mass in the moving system through differentiation:

$$\xi = l \sin \varphi, \quad \eta = -l \cos \varphi,$$

$$\dot{\xi} = l \dot{\varphi} \cos \varphi, \quad \dot{\eta} = l \dot{\varphi} \sin \varphi,$$

$$\ddot{\xi} = l \ddot{\varphi} \cos \varphi - l \dot{\varphi}^2 \sin \varphi, \quad \ddot{\eta} = l \ddot{\varphi} \sin \varphi + l \dot{\varphi}^2 \cos \varphi.$$
This yields the relative acceleration
\[ \mathbf{a}_r = \dot{\xi} \mathbf{e}_\xi + \ddot{\eta} \mathbf{e}_\eta = (l\ddot{\varphi} \cos \varphi - l\dot{\varphi}^2 \sin \varphi)\mathbf{e}_\xi + (l\ddot{\varphi} \sin \varphi + l\dot{\varphi}^2 \cos \varphi)\mathbf{e}_\eta. \]

Introduction into the equation of motion leads to the components of the equation of motion in the direction of the axes \( \xi \) and \( \eta \):
\[ m (l\ddot{\varphi} \cos \varphi - l\dot{\varphi}^2 \sin \varphi) = -S \sin \varphi - ma_0, \]
\[ m (l\ddot{\varphi} \sin \varphi + l\dot{\varphi}^2 \cos \varphi) = S \cos \varphi - mg. \]
These are two equations for the unknowns \( \varphi \) and \( S \). Solving for \( \varphi \) yields the equation of motion
\[ l\ddot{\varphi} + g \sin \varphi + a_0 \cos \varphi = 0. \]
Note that the position \( \varphi_0 = -\arctan \frac{a_0}{g} \) is obtained for \( \ddot{\varphi} = 0 \).
The pendulum oscillates about this position for \( \ddot{\varphi} \neq 0 \).
Example 6.6  The two disks in Fig. 6.9 rotate with constant angular velocities $\Omega$ and $\omega$ about their respective axes.

Determine the absolute acceleration of point $P$ at the instant shown.

Solution  We describe the motion of point $P$ in a coordinate system $x, y, z$ which is fixed to the large disk. The absolute acceleration of $P$ is

$$a_P = a_f + a_r + a_c,$$

where the acceleration of the reference frame and the relative acceleration are given by

$$a_f = \begin{bmatrix} 0 \\ -(a + r \cos \varphi)\Omega^2 \\ 0 \end{bmatrix}, \quad a_r = \begin{bmatrix} 0 \\ -r \omega^2 \cos \varphi \\ -r \omega^2 \sin \varphi \end{bmatrix}.$$

We also write the angular velocity of the reference frame and the relative velocity as column vectors:

$$\Omega = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix}, \quad v_r = \begin{bmatrix} 0 \\ -r \omega \sin \varphi \\ r \omega \cos \varphi \end{bmatrix}.$$
and we calculate the Coriolis acceleration:

$$a_c = 2\Omega \times v_r \Rightarrow a_c = \begin{bmatrix} 2r\omega \sin \varphi \\
0 \\
0 \end{bmatrix}.$$

Combining yields

$$a_P = \begin{bmatrix} 2r\omega \sin \varphi \\
-(\alpha + r \cos \varphi)\Omega^2 - r\omega^2 \cos \varphi \\
-r\omega^2 \sin \varphi \end{bmatrix}.$$
Example 6.7  A horizontal circular platform (radius \( r \)) rotates with constant angular velocity \( \Omega \) (Fig. 6.10). A block (mass \( m \)) is locked in a frictionless slot at a distance \( a \) from the center of the platform. At time \( t = 0 \) the block is released.

Determine the velocity \( v_r \) of the block relative to the platform when it reaches the rim of the platform.

Solution  We describe the motion of the block in a coordinate system \( x, y \) which is fixed to the platform. The absolute acceleration of the block is given by

\[
\mathbf{a}_B = \mathbf{a}_f + \mathbf{a}_r + \mathbf{a}_c.
\]

Here, the acceleration of the reference frame, the relative acceleration and the Coriolis acceleration are

\[
\mathbf{a}_f = \begin{bmatrix} -x \Omega^2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_c = \begin{bmatrix} 0 \\ 2 \Omega \dot{x} \end{bmatrix}.
\]

Thus, the absolute acceleration becomes

\[
\mathbf{a}_B = \begin{bmatrix} \ddot{x} - \Omega^2 x \\ 2 \Omega \dot{x} \end{bmatrix}.
\]

The equation of motion for the block is

\[
m \mathbf{a}_B = \mathbf{F}
\]
where the force $F$ (which is exerted from the slot on the block) is

$$F = \begin{bmatrix} 0 \\ F_y \end{bmatrix}.$$ 

We now write down the $x$-component of the equation of motion:

$$m(\ddot{x} - \Omega^2 x) = 0 \quad \rightarrow \quad \ddot{x} - \Omega^2 x = 0.$$ 

The general solution of this differential equation is given by

$$x(t) = A \cosh \Omega t + B \sinh \Omega t.$$ 

With the initial conditions

$$x(0) = a \quad \rightarrow \quad A = a,$$

$$\dot{x}(0) = 0 \quad \rightarrow \quad B = 0$$

we obtain

$$x(t) = a \cosh \Omega t.$$ 

When the block reaches the rim of the platform, the condition

$$x(t_R) = r \quad \rightarrow \quad \cosh \Omega t_R = r/a$$

is satisfied. Thus the relative velocity is (note that $\cosh^2 x - \sinh^2 x = 1$)

$$\dot{x}(t_R) = a \Omega \sinh \Omega t_R \quad \rightarrow \quad \dot{x}(t_R) = \Omega \sqrt{r^2 - a^2}.$$
Example 6.8 A simple pendulum is attached to point 0 of a circular disk (Fig. 6.11). The disk rotates with a constant angular velocity $\Omega$; the pendulum oscillates in the horizontal plane.

Determine the circular frequency of the oscillations. Assume small amplitudes and neglect the weight of the mass.

Solution We introduce a rotating $\xi, \eta, \zeta$-coordinate system. Then

$$\Omega = \Omega e_\zeta, \quad a_0 = \ddot{r}_0 = -r\Omega^2 e_\xi,$$

$$\dot{\Omega} = 0, \quad r_{0P} = l \cos \varphi e_\xi + l \sin \varphi e_\eta.$$

The relative velocity can be expressed by the relative angular velocity $\varphi^*$ ($^*$: time derivative relative to the moving frame):

$$v_r = l \varphi^* \quad \rightarrow \quad v_r = -l \varphi^* \sin \varphi e_\xi + l \varphi^* \cos \varphi e_\eta.$$

Thus, the fictitious forces $F_f$ and $F_c$ are

$$F_f = -ma_0 - m\Omega \times (\Omega \times r_{0P}) = m(rl^2 + l\Omega^2 \cos \varphi)e_\xi + m\Omega^2 l \sin \varphi e_\eta,$$

$$F_c = -2m\Omega \times v_r = 2mll^2 \varphi^* (e_\xi \cos \varphi + e_\eta \sin \varphi).$$
With the tangential relative acceleration $a_{\tau} = l \dot{\varphi}^{**}$ the equation of motion in the tangential direction is obtained as:

\[ m l \ddot{\varphi}^{**} = m (l \Omega^2 + 2 l \dot{\varphi} \Omega) \sin \varphi \cos \varphi \]

\[ - m r \Omega^2 + (l \Omega^2 + 2 l \dot{\varphi} \Omega) \cos \varphi \sin \varphi = - m r \Omega^2 \sin \varphi. \]

We assume small amplitudes ($\sin \varphi \approx \varphi$). This yields

\[ \ddot{\varphi}^{**} + \frac{r \Omega^2}{l} \varphi = 0. \]

Hence, the circular frequency of the oscillations is

\[ \omega = \sqrt{\frac{r}{l} \Omega}. \]
Example 6.9 A drum rotates with angular velocity $\omega$ about point $B$ (Fig. 6.12). Pin $C$ is fixed to the drum; it moves in the slot of link $AD$.

Determine the angular velocity $\omega_{AD}$ of link $AD$ and the velocity $v_r$ of the pin relative to the link at the instant shown.

Solution We use the rotating coordinate system $x, y$ as shown in the figure. The (absolute) velocity of pin $C$ is given by

$$v_C = 3l\omega \begin{bmatrix} \cos \beta \\ -\sin \beta \end{bmatrix}.$$ 

With the geometrical relations

$$a = \sqrt{16l^2 + 9l^2} = 5l, \quad \sin \beta = \frac{3}{5}, \quad \cos \beta = \frac{4}{5},$$

we can write

$$v_C = \frac{3l\omega}{5} \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$ 

The velocity of the reference frame at point $C$ and the velocity of pin $C$ relative to the moving frame are

$$v_f = \dot{\beta}l \begin{bmatrix} 0 \\ \frac{3}{5} \end{bmatrix}, \quad v_r = v_r \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ 

With

$$v_C = v_f + v_r \rightarrow \frac{3l\omega}{5} \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \dot{\beta}l \begin{bmatrix} 0 \\ \frac{3}{5} \end{bmatrix} + v_r \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

we finally obtain

$$\omega_{AD} = \dot{\beta} = -\frac{9\omega}{25}, \quad v_r = \frac{12}{5}l\omega \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
Example 6.10  A point $P$ moves along a circular path (radius $r$) on a platform with a constant relative velocity $v_r$ (Fig. 6.13). The platform rotates with a constant angular velocity $\omega$ about point $A$. The eccentricity $e$ is given.

Determine the relative, fixed frame-, Coriolis, and absolute accelerations of $P$.

Solution  We introduce the coordinate system $\zeta, \eta, \zeta$ as shown in the figure. Its origin is located at the center 0 of the platform; it rotates with the platform. Thus point $P$ undergoes a circular motion relative to this system.

With the magnitude $a_r = v_r^2/r$ of the relative acceleration and its direction (from $P$ to 0) we can write

$$ a_r = -\frac{v_r^2}{r}(e_\zeta \cos \varphi + e_\eta \sin \varphi). $$

With

$$ \omega = \omega e_\zeta, \quad \dot{\omega} = 0, \quad r_{0P} = e_\zeta r \cos \varphi + e_\eta r \sin \varphi, $$

$$ v_r = v_r(-e_\zeta \sin \varphi + e_\eta \cos \varphi), \quad \dot{r}_0 = a_0 = -e\omega^2 e_\zeta $$

we obtain

$$ a_f = a_0 + \omega \times (\omega \times r_{0P}) $$

$$ = -e\omega^2 e_\zeta + r\omega[ e_\zeta \times (e_\zeta \times e_\zeta \cos \varphi) + e_\zeta \times (e_\zeta \times e_\eta \sin \varphi)] $$

$$ = (e + r \cos \varphi)\omega^2 e_\zeta - r\omega^2 \sin \varphi e_\eta, $$

$$ a_c = 2\omega \times v_r = 2\omega v_r(e_\zeta \times (-e_\zeta \sin \varphi) + e_\zeta \times e_\eta \cos \varphi) $$

$$ = -2\omega v_r(e_\zeta \cos \varphi + e_\eta \sin \varphi). $$
Thus, the absolute acceleration is found as

\[ a = a_f + a_r + a_c \]

\[ = -\left[ \omega^2 + \left( r\omega^2 + \frac{v_r^2}{r} + 2\omega v_r \right) \cos \varphi \right] e_\xi - \left[ r\omega^2 + \frac{v_r^2}{r} + 2\omega v_r \right] \sin \varphi e_\eta \]

\[ = -\left[ \omega^2 + r(\omega + \frac{v_r}{r})^2 \cos \varphi \right] e_\xi - r(\omega + \frac{v_r}{r}) \sin \varphi e_\eta . \]
Example 6.11 A circular ring (radius $r$) rotates with constant angular velocity $\Omega$ about the $x$-axis (Fig. 6.14). A point mass $m$ moves without friction inside the ring.

Derive the equations of motion and determine the equilibrium positions of the point mass relative to the ring.

Solution We describe the motion of the point mass in the $x, y, z$-coordinate system (see the figure) which rotates with the ring. The absolute acceleration $\mathbf{a}$ of the point mass is given by

$$\mathbf{a} = a_f + a_r + a_c,$$

where the acceleration of the reference frame and the relative acceleration are

$$a_f = \begin{bmatrix} 0 \\ -r\Omega^2 \sin \varphi \\ 0 \end{bmatrix}, \quad a_r = \begin{bmatrix} -r\dot{\varphi} \cos \varphi - r\ddot{\varphi} \sin \varphi \\ r\dot{\varphi} \sin \varphi + r\ddot{\varphi} \cos \varphi \\ 0 \end{bmatrix}.$$

With the angular velocity of the ring and the relative velocity

$$\Omega = \begin{bmatrix} \Omega \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_r = \begin{bmatrix} -r\dot{\varphi} \sin \varphi \\ r\dot{\varphi} \cos \varphi \\ 0 \end{bmatrix},$$

we can calculate the Coriolis acceleration

$$a_c = 2\Omega \times \mathbf{v}_r \quad \Rightarrow \quad a_c = \begin{bmatrix} 0 \\ 0 \\ 2r\Omega \dot{\varphi} \cos \varphi \end{bmatrix}.$$

The equation of motion is given by

$$m\mathbf{a} = \mathbf{W} + \mathbf{N},$$
where
\[
W = \begin{bmatrix} -mg \\ 0 \\ 0 \end{bmatrix}
\]
is the weight of the point mass and
\[
N = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}
\]
with \( N_y/N_x = \tan \varphi \)
is the force exerted from the ring on the point mass. Now, we can write down the components of the equation of motion:
\[
-m(r\dot{\varphi}^2 \cos \varphi + r\ddot{\varphi} \sin \varphi) = -mg + N_x, \\
-m(r\dot{\varphi}^2 \sin \varphi - r\ddot{\varphi} \cos \varphi + r\Omega^2 \sin \varphi) = N_x \tan \varphi, \\
2mr\Omega \ddot{\varphi} \cos \varphi = N_z.
\]
Positions of equilibrium relative to the ring are characterized by
\[
\dot{\varphi} = 0, \quad \ddot{\varphi} = 0 \quad \Rightarrow \quad (r\Omega^2 + \frac{g}{\cos \varphi}) \sin \varphi = 0.
\]
This yields
\[
\varphi_1 = 0, \quad \varphi_2 = \pi, \quad \varphi_{3,4} = \pi \pm \arccos \left( \frac{g}{r\Omega^2} \right).
\]
The positions \( \varphi_{3,4} \) exist only for \( \Omega^2 > g/r \).
Example 6.12 A point \( P \) moves on a square plate along a circular path (radius \( r \)) with a constant relative velocity \( v_r \). The plate moves horizontally with the constant acceleration \( a_0 \) (Fig. 6.15).

Determine the magnitude of the absolute acceleration of \( P \).

**Solution** The components of the relative velocity in the moving reference frame \( \xi, \eta \) are

\[
\begin{align*}
v_r\xi &= \xi^* = -v_r \sin \varphi, \\
v_r\eta &= \eta^* = v_r \cos \varphi
\end{align*}
\]

(\( \ast \): time derivative relative to the moving frame). Differentiation with respect to the moving frame leads to the components of the relative acceleration (note: \( r\varphi = v_r \)):

\[
\begin{align*}
a_r\xi &= \xi^{**} = -v_r v_\varphi \cos \varphi = -\frac{v_r^2}{r} \cos \varphi, \\
a_r\eta &= \eta^{**} = -v_r v_\varphi \sin \varphi = -\frac{v_r^2}{r} \sin \varphi.
\end{align*}
\]

Since the reference frame undergoes a translation, the absolute acceleration is given by

\[
\begin{align*}
a_x &= a_0 + a_r\xi = a_0 - \frac{v_r^2}{r} \cos \varphi, \\
a_y &= a_r\eta = -\frac{v_r^2}{r} \sin \varphi.
\end{align*}
\]

It has the magnitude

\[
a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_0^2 + \frac{v_r^4}{r^2} - 2a_0 \frac{v_r^2}{r} \cos \varphi}.
\]
**Example 6.13** A crane starts to move from rest with a constant acceleration \( a_0 \) along a straight track. At the same time, the jib begins to rotate with constant angular velocity \( \omega \), and the trolley on the jib begins to move towards point 0 with constant relative acceleration \( b_c \) (Fig. 6.16). The initial positions of the jib and of the trolley are given by \( \varphi_0 \) and \( s_0 \).

Determine the absolute velocity and the absolute acceleration of the trolley as functions of the time \( t \).

**Solution** We use the fixed coordinate system \( x, y, z \), where the \( x \)-axis coincides with the track. In addition, we introduce the rotating coordinate system \( \xi, \eta, \zeta \), where the \( \xi \)-axis rotates with the jib. Then, the general equations for the absolute velocity and the absolute acceleration are

\[
\mathbf{v} = \mathbf{v}_0 + \mathbf{\omega} \times \mathbf{r}_{0P} + \mathbf{v}_r,
\]

\[
\mathbf{a} = \mathbf{a}_0 + \mathbf{\omega} \times \mathbf{r}_{0P} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}_{0P}) + 2 \mathbf{\omega} \times \mathbf{v}_r + \mathbf{a}_r.
\]

We measure the time \( t \) from the beginning of the motion. Then, making use of the given accelerations, angular velocity and initial conditions, we obtain

\[
\mathbf{a}_0 = b_0 e_z \quad \Rightarrow \quad \mathbf{v}_0 = b_0 t e_x,
\]

\[
\mathbf{\omega} = \omega e_\zeta, \quad \dot{\mathbf{\omega}} = 0,
\]

\[
\mathbf{a}_r = -b_c e_\xi \quad \Rightarrow \quad \mathbf{v}_r = -b_c t e_\xi \quad \Rightarrow \quad \mathbf{r}_{0P} = (-\frac{1}{2} b_c t^2 + s_0) e_\xi,
\]

and

\[
\mathbf{\omega} \times \mathbf{r}_{0P} = \mathbf{\omega} (\frac{1}{2} b_c t^2 + s_0) e_\eta, \quad 2 \mathbf{\omega} \times \mathbf{v}_r = -2 \mathbf{\omega} b_c t e_\eta,
\]

\[
\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}_{0P}) = -\mathbf{\omega}^2 (-\frac{1}{2} b_c t^2 + s_0) e_\xi.
\]
With the relation \( e_x = e_\xi \cos \varphi - e_\eta \sin \varphi \), where \( \varphi = \varphi_0 + \omega t \), we finally obtain

\[
\begin{align*}
v &= [b_0 t \cos \varphi - b_c t] e_\xi + [-b_0 t \sin \varphi + \omega (\frac{1}{2} b_c t^2 + s_0)] e_\eta, \\
a &= [b_0 \cos \varphi - \omega^2 (-\frac{1}{2} b_c t^2 + s_0) - b_c] e_\xi - [b_0 \sin \varphi + 2 \omega b_c t] e_\eta.
\end{align*}
\]